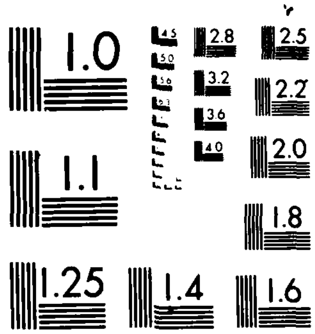
AD-A176 480 THE CHARACTERISTICS OF THE SECOND-ORDER TARGET MODEL (U) ROYAL AIRCRAFT ESTABLISHMENT FARNBOROUGH (ENGLAND) D J SALMOND AUG 85 RAE-TR-85071 DRIC-BR-99021 1/1

UNCLASSIFIED F/G 19/5 NL

END
DATE
FILMED
-87



MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS 1963 A

UNLIMITED

BR99021 (2)

TR 85071

REPRINT

TR 85071

AD-A176 480

ROYAL AIRCRAFT ESTABLISHMENT

Technical Report 85071

August 1985

THE CHARACTERISTICS OF THE SECOND-ORDER TARGET MODEL

by

D. J. Salmond

Procurement Executive, Ministry of Defence
Farnborough, Hants

UNLIMITED

UDC 519.233.2 : 519.271.01 : 519.217 : 517.97 : 517.952

ROYAL AIRCRAFT ESTABLISHMENT

Technical Report 85071

Received for printing 8 August 1985

THE CHARACTERISTICS OF THE SECOND-ORDER TARGET MODEL

by

D. J. Salmond

SUMMARY

Statistics of the trajectories generated by a second-order target model are presented, and using these statistics the distribution of the change in target heading over any time period is derived. This distribution is the basis of a new technique for choosing the model manoeuvre parameter which is suitable for a given class of targets. Design curves for this technique are supplied.

Departmental Reference: AW 18

LIST OF CONTENTS

1 INTRODUCTION

An essential component of most target tracking filters is a mathematical model describing the kinematics of the target. The target model represents the level of manoeuvre which the target of interest is expected to exhibit, and is used in the tracking filter for smoothing the estimate of the target state (*eg* position and velocity). This Report describes the properties of a simple linear second-order target model, which is widely used in tracking filters[1]. The chief objectives of this Report are to present the statistics of the target trajectories described by the target model and using these statistics to derive a new method for selecting the manoeuvre parameter of the model appropriate to a given class of targets. The trajectory statistics are also used to obtain a search region, such that there is a given probability of a target which obeys the second-order model being within the region at some time after observing the target state (*ie* assessing how quickly targetting information becomes stale).

The continuous version of second-order target model is derived under the assumption that the acceleration of the target in each Cartesian co-ordinate is an independent Gaussian white noise process. The discrete version of the model is similar except that the target acceleration is taken to be a Gaussian white sequence, the acceleration being assumed constant over each time step. The model has the advantage of simplicity while providing, for many applications, quite an adequate representation of target trajectories (in two or three dimensions). The trajectory described by the model is a variation about a constant velocity course, whose magnitude and direction are defined by initial conditions. The deviation from this mean course is controlled by a single parameter, the power spectral density of the Gaussian driving noise (this parameter is taken to be the same for each Cartesian co-ordinate in this Report). If this parameter is zero, then the target trajectory is a constant velocity course.

The main contribution of this Report is the derivation of the probability density function (pdf) of the change in heading angle (*ie* the change in direction of the velocity vector) in a given period, for a target whose motion is governed by the second-order model. This pdf is a function of the model manoeuvre parameter, the initial speed of the target and the time period over which the heading change is considered, and it provides a useful measure of the degree of target manoeuvrability which is being modelled. Hence this distribution of heading change forms the basis of a new method for selecting the manoeuvre parameter appropriate to a given class of targets (section 4). In section 3 the problem of defining a search region for a target obeying the second-order model and given prior imperfect targetting information is discussed. Assuming the manoeuvre parameter of the target is known, it is shown that a suitable search area for the two-dimensional case is an ellipse (or an ellipsoid for three dimensions) centred on the expected position of the target. The search region and the distribution of heading angle change are both derived from the conditional pdf of target state, which follows directly from well-known general results for linear systems driven by Gaussian noise. These general results are collected in Appendix A, which also discusses the relationship between linear systems and the Markov property.

Save for the derivation of the model in section 2, the chief emphasis of this Report is on the discrete version of the target model, since nowadays tracking filters are usually implemented digitally. However at the end of sections 3 and 4 and Appendix A, the chief results are also presented for the continuous version of the target model.

2 DERIVATION OF THE TARGET MODEL

The continuous version of the target model is based on the premise that the acceleration of the target is a zero mean, Gaussian, white noise process. Hence each Cartesian co-ordinate x may be described by*

$$\ddot{x} = w' , \tag{1}$$

where w' is a Gaussian noise process with

$$\left.\begin{aligned} E[w'] &= 0 \\ \text{and} \qquad E[w'(t+\tau)w'(t)] &= q'\delta(\tau) \end{aligned}\right\} , \tag{2}$$

where E is the expectation operator, $\delta(\tau)$ is the Dirac delta function and q' is the power spectral density (constant) of the acceleration noise. The white noise process w' may be viewed as the limit of a bandlimited Gaussian noise process with a constant power density spectrum q' up to the cut-off frequency ω_c . The autocorrelation function of such a bandlimited process is

$$E[w'(t+\tau)w'(t)] = q'\frac{\sin \omega_c\tau}{\pi\tau}$$

and as $\omega_c \to \infty$, so the noise becomes white,

$$\frac{\sin \omega_c\tau}{\pi\tau} \to \delta(\tau)$$

(see Papoulis[10], p 98), from which equation (2) follows.

The general state space representation of a continuous linear system driven by Gaussian white noise is as follows:

$$\dot{\underline{x}} = F\underline{x} + G\underline{w}' , \tag{3}$$

where $\underline{x}$ is the state vector, F and G are matrices which may be time-dependent and

---

* A rigorous statement requires the use of Ito stochastic calculus, in which case equation (1) would be written

$$d\dot{x} = du ,$$

where du is a Wiener process with zero mean and variance q'dt (see Sage and Melsa[4] or Jazwinsky[9] for further details).

$\underline{w}'$ is a vector of white Gaussian noise (sometimes called process noise). In this Report it is assumed that

$$\left.\begin{aligned} E[\underline{w}'] &= 0 \\ \text{and} \qquad E[\underline{w}'(t+\tau)\underline{w}'^T(t)] &= Q'\delta(\tau) \end{aligned}\right\} , \tag{4}$$

where Q' is the power spectral density matrix of the process noise.

The target model can be put into this general form by setting

$$\left.\begin{aligned} \underline{x} &= \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \\ \underline{w}' &= w' \\ F &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ G &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \text{and} \qquad Q' &= q' . \end{aligned}\right\} \tag{5}$$

The state transition matrix (see Ref 5) of this system is

$$\Phi'(t+\tau,t) = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

which depends only on the time interval τ . It can be seen that the target velocity $\dot{x}$ is the integral of w' and is therefore a Wiener process (see Papoulis[2]).

For the discrete version of the target model, the target acceleration is assumed to be a white Gaussian sequence, the acceleration being constant over a time step. Hence, integrating over a time step Δt , for each Cartesian co-ordinate,

$$\left.\begin{aligned} x\big((k+1)\Delta t\big) &= x(k\Delta t) + \Delta t\dot{x}(k\Delta t) + \tfrac{1}{2}\Delta t^2 w_k \\ \text{and} \qquad \dot{x}\big((k+1)\Delta t\big) &= \dot{x}(k\Delta t) + \Delta t w_k \end{aligned}\right\} , \tag{6}$$

where $\{w_k\}$ is a white Gaussian sequence of variance q , *ie*

$$E[w_k] = 0$$

and

$$E\{w_k w_j\} = q\delta_{kj} \quad .$$

The general state space representation of a discrete linear system driven by a Gaussian white noise sequence is

$$\underline{x}_{k+1} = \phi_k \underline{x}_k + \Gamma \underline{w}_k \quad , \tag{7}$$

where $\underline{x}_k$ is the state vector at time $k\Delta t$,
ϕ_k is the state transition matrix from k to $k+1$,
Γ is a matrix
and $\{\underline{w}_k\}$ is a random Gaussian sequence with

$$E[\underline{w}_k] = 0$$

and $$E\left[\underline{w}_k \underline{w}_j^T\right] = Q\delta_{kj} \quad ,$$

where Q is the covariance of the sequence $\{\underline{w}_k\}$.

The discrete target model given by (6) can be written in this general form as follows:

$$\underline{x}_{k+1} = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \underline{x}_k + \begin{pmatrix} \Delta t^2/2 \\ \Delta t \end{pmatrix} w_k \tag{8}$$

where $\underline{x}_k = \underline{x}(k\Delta t) = \begin{pmatrix} x(k\Delta t) \\ \dot{x}(k\Delta t) \end{pmatrix}$

and $Q = q$.

For this model, the target velocity $\dot{x}(k\Delta t)$ is a random walk (see Jenkins and Watts[3]).

If the value of q is chosen so that

$$q = \frac{q'}{\Delta t} \quad ,$$

it can be shown that as $\Delta t \to 0$ and the samples became dense, the discrete target model (8) tends to the continuous model (5). To see this, rewrite equation (8) as

$$\frac{1}{\Delta t}(\underline{x}_{k+1} - \underline{x}_k) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{x}_k + \begin{pmatrix} \Delta t/2 \\ 1 \end{pmatrix} w_k$$

and let $\Delta t \to 0$, $k \to \infty$ and $j \to \infty$ such that $k\Delta t = t$ and $j\Delta t = \tau$. In this case it can be shown that (see Sage and Melsa[4]),

$$E[w_k w_j] = q\delta_{kj} = \frac{q'}{\Delta t}\delta_{kj} \rightarrow q'\delta(t-\tau)$$

and

$$\dot{\underline{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w'$$

which is the continuous model.

3 THE PROBABILITY DENSITY FUNCTION OF THE TARGET STATE

3.1 The discrete case

3.1.1 Propagation of the mean and covariance of the state vector

The state space representation of the target model for discrete time steps is given by equation (8), and from Appendix A, section A.1 it can be seen that the sequence of state vectors resulting from this model is a homogeneous Gauss-Markov sequence. Suppose it is known that the pdf of $\underline{x}_k$ is Gaussian with mean $\bar{\underline{x}}_k$ and covariance M_k, but that $\underline{x}_k$ is unknown. From equations (A-9) and (A-10) of Appendix A, section A.1, it can be shown that, after an interval of $\ell\Delta t$, the pdf of $\underline{x}_{k+\ell}$ conditioned on this information, *ie* $p(\underline{x}_{k+\ell}|\bar{\underline{x}}_k, M_k)$, is Gaussian with mean

$$\begin{aligned} \bar{\underline{x}}_{k+\ell} &= \phi^{\ell}\bar{\underline{x}}_k \\ &= \begin{pmatrix} \bar{x}_k + (\ell\Delta t)\bar{\dot{x}}_k \\ \bar{\dot{x}}_k \end{pmatrix} \end{aligned} \tag{9}$$

and covariance

$$M_{k+\ell} = \phi^{\ell} M_k \phi^{\ell^T} + q(\Delta t)^2 \sum_{j=0}^{\ell-1} \begin{pmatrix} \left[\frac{\Delta t}{2}\left(1 + 2(\ell-j-1)\right)\right]^2 & \frac{\Delta t}{2}\left(1+2(\ell-j-1)\right) \\ \frac{\Delta t}{2}\left(1+2(\ell-j-1)\right) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (m_{11})_k + 2\ell\Delta t(m_{12})_k + (\ell\Delta t)^2(m_{22})_k + \frac{q(\Delta t)^4}{12}(4\ell^2-1)\ell & (m_{12})_k + \ell\Delta t(m_{22})_k + \frac{q(\Delta t)^3\ell^2}{2} \\ (m_{12})_k + \ell\Delta t(m_{22})_k + \frac{q(\Delta t)^3\ell^2}{2} & (m_{22})_k + q(\Delta t)^2\ell \end{pmatrix}, \tag{10}$$

where $\begin{pmatrix} (m_{11})_k & (m_{12})_k \\ (m_{12})_k & (m_{22})_k \end{pmatrix} = M_k$.

The element m_{11} of the covariance matrix is the variance of the target position x , m_{22} is the variance of the target velocity $\dot{x}$ and m_{12} is the correlation between the target's position and velocity. Thus equations (9) and (10) describe the propagation of the mean and the covariance of the target state vector given that the state vector $\underline{x}_k$ has a mean of $\bar{\underline{x}}_k$ and a covariance matrix M_k . Also with $\ell = 1$, equations (9) and (10) are implicit in the $\alpha - \beta$ tracking filter (in which each target co-ordinate is filtered separately), equation (10) giving the covariance of the target state immediately before a measurement is taken.

If the position x_k and velocity $\dot{x}_k$ of the target are known perfectly, then the conditional pdf $p(\underline{x}_{k+\ell}|\underline{x}_k)$ is Gaussian with mean

$$\begin{pmatrix} x_k + (\ell\Delta t)\dot{x}_k \\ \dot{x}_k \end{pmatrix} \tag{11}$$

and covariance

$$q(\Delta t)^2\ell \begin{pmatrix} \frac{(\Delta t)^2(4\ell^2 - 1)}{12} & \frac{\ell\Delta t}{2} \\ \frac{\ell\Delta t}{2} & 1 \end{pmatrix} . \tag{12}$$

These relations may be derived from (9) and (10) by letting $M_k \to 0$ so that $\bar{\underline{x}}_k \to \underline{x}_k$, or directly from (A-5) and (A-6) of Appendix A, section A.1. If the initial target state $\underline{x}_0$ is known perfectly, then the expected path of the target is a constant velocity trajectory (see (11)) and the probable deviation from this path is given by the conditional covariance (12). It can be seen that the conditional covariance (12) is only dependent on the interval ℓ , and the conditional variance of position increases approximately with $(\ell\Delta t)^3$, while that of velocity increases with $(\ell\Delta t)$.

3.1.2 Likelihood ellipses of the Gaussian pdf of the state vector

The propagation of the pdf of $\underline{x}$ may be visualized in the state space $(x,\dot{x})$ by the evolution of contours of constant probability density or likelihood. For the Gaussian pdf of $\underline{x}$, these contours are ellipses and are known as likelihood ellipses. The centres of these evolving ellipses lie on the mean state space trajectory which is a constant velocity path represented by a straight line parallel to the x-axis in $(x,\dot{x})$ space. Suppose that the initial state vector $\underline{x}_0$ is Gaussian distributed with covariance M_0 . If the position is initially uncorrelated with the velocity, *ie* $(m_{12})_0 = 0$, then the likelihood ellipses corresponding to $\underline{x}_0$ will have their principal axes aligned with the x and $\dot{x}$ axes of the state space. Now suppose that there is no system driving noise

($q = 0$) so that the target trajectories will have constant velocities equal to their initial velocities. In this case the likelihood ellipses at a later time $\ell\Delta t$ may be obtained via a linear shear transform of the initial ellipse:

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix} \rightarrow \begin{pmatrix} x + (\ell\Delta t)\dot{x} \\ \dot{x} \end{pmatrix} .$$

Thus the likelihood ellipses will become slewed with increasing time (see Fig 1) as targets with initial velocities greater than the mean will travel further than the mean target, while targets with initial velocities less than the mean will travel less far. In other words, the correlation between the position and velocity increases with time and this effect is described mathematically by the first term in the covariance propagation equation (10), *ie*

$$M_\ell = \phi^\ell M_0 \phi^{\ell^T} .$$

If the system driving noise is non-zero (*ie* $q > 0$), then the target trajectories are no longer constant velocity paths and this introduces further uncertainty into the target state with increasing time, so increasing the area of the ellipses (see Fig 2). This is represented by the second term of equation (10), which is superposed with the first effect.

If the initial target state vector is known perfectly (*ie* $M_0 = 0$), then as already discussed, the propagation oi the mean and covariance of the Gaussian pdf of the state vector is given by equations (11) and (12). For this case, in Figs 3 to 7, the evolution of the likelihood ellipses has been shown for several different values of the acceleration noise variance q (the manoeuvre parameter). In these diagrams, the velocity axis has been normalized to the mean velocity v_0 (which is also the initial target velocity) and the distance axis has been noramlized to $v_0\Delta t$, the mean distance travelled in one time step. The initial state is set to

$$x_0 = 0$$

$$\dot{x}_0 = v_0$$

and q is varied between $10^{-5}(v_0/\Delta t)^2$ and $10^{-1}(v_0/\Delta t)^2$, the value increasing by a factor of 10 in each Figure. In each case likelihood ellipses of the state vector are shown for $\ell = 10, 30, 60$ and 100. The ellipses have been chosen so that there is a 95% probability of the state falling inside the ellipse. Also shown in each Figure is a realization of the state space trajectory generated by driving equation (8) with a random Gaussian sequence of variance q, and the four points marked by squares on the trajectory are the states corresponding to the four likelihood ellipses. In almost every case these points are within the 95% likelihood ellipses.

3.1.3 Search regions for the second-order target model

It is often necessary to define a search region for a target based on prior imperfect targetting information. If the target obeys the second-order model given by equation (8), and the uncertainty in the targetting information for each Cartesian co-ordinate is independent and can be expressed in terms of the covariance matrix M_k (which may be different for each co-ordinate), then the pdf of the target position, at a time $\ell\Delta t$ after the observation, is Gaussian with each co-ordinate independent and having a mean given by (9) and a variance $(m_{11})_\ell$ given by (10). If the search region is defined as that region for which there is a given probability of the target being within the region, and for which the value of the position pdf at every point within the region exceeds the value of the pdf outside the region, then in two dimensions the search area is bounded by a likelihood ellipse and in three dimensions it is bounded by an ellipsoid (see Bryson and Ho[5]), with the expected target position at the centre of the region. In two dimensions, the search area is defined by

$$\frac{(x - \bar{x})^2}{m_{11_x}} + \frac{(y - \bar{y})^2}{m_{11_y}} = f_2(x,y) < T_2 \quad ,$$

where $\bar{x} = \bar{x}_0 + \ell\Delta t\bar{\dot{x}}_0$ from (9),

$$m_{11_x} = (m_{11_x})_0 + 2\ell\Delta t(m_{12_x})_0 + (\ell\Delta t)^2(m_{22_x})_0 + \frac{q(\Delta t)^4}{12}(4\ell^2 - 1)\ell \quad \text{from (10)},$$

$\begin{pmatrix} \bar{x}_0 \\ \bar{\dot{x}}_0 \end{pmatrix}$ is the observation of the x co-ordinate position and velocity,

$\begin{pmatrix} (m_{11_x})_0 & (m_{12_x})_0 \\ (m_{12_x})_0 & (m_{22_x})_0 \end{pmatrix}$ is the covariance of the error in this observation,

and $\bar{y}$ and m_{11_y} are similar.

Likewise, in three dimensions the search volume is defined by

$$\frac{(x - \bar{x})^2}{m_{11_x}} + \frac{(y - \bar{y})^2}{m_{11_y}} + \frac{(z - \bar{z})^2}{m_{11_z}} = f_3(x,y,z) < T_3 \quad .$$

Since $f_2(x,y)$ is the sum of the squares of two independent Gaussian variates, it has a χ^2 distribution with 2 degrees of freedom, and similarly $f_3(x,y,z)$ has a χ^2 distribution with 3 degrees of freedom. Hence if the required probability of the target being within the search region is P, then the thresholds T_2 and T_3 may be found from tables of the χ^2 distribution. For example, for $P = 0.95$, $T_2 = 6.00$ and $T_3 = 7.81$.

3.2 The continuous case

In this brief section, the equivalent continuous versions of the major results of section 3.1 are given. These results may be derived by setting $q = q'/\Delta t$ and letting $\Delta t \to 0$ and $\ell \to \infty$ so that $\ell\Delta t \to t$.

The stochastic process $\underline{x}(t)$ generated by the continuous target model (5) is a homogeneous Gauss-Markov process. If at time t , the pdf of $\underline{x}(t)$ is Gaussian with mean $\bar{\underline{x}}(t)$ and covariance $M(t)$, then the pdf of $\underline{x}(t+\tau)$ conditioned on this information, *ie* $p\{\underline{x}(t+\tau)|\bar{\underline{x}}(t),M(t)\}$, is Gaussian with mean

$$\bar{\underline{x}}(t+\tau) = \begin{pmatrix} \bar{x}(t) + \tau\bar{\dot{x}}(t) \\ \bar{\dot{x}}(t) \end{pmatrix}$$

and covariance

$$M(t+\tau) = \begin{pmatrix} m_{11}(t) + 2\tau m_{12}(t) + \tau^2 m_{22}(t) + q'\dfrac{\tau^3}{3} & m_{12}(t) + \tau m_{22}(t) + q'\dfrac{\tau^2}{2} \\ m_{12}(t) + \tau m_{22}(t) + q'\dfrac{\tau^2}{2} & m_{22}(t) + q'\tau \end{pmatrix} ,$$

where $M(t) = \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{12}(t) & m_{22}(t) \end{pmatrix}$.

Also if $\underline{x}(t)$ is known, then the conditional pdf $p\{\underline{x}(t+\tau)|\underline{x}(t)\}$ is Gaussian with mean

$$\begin{pmatrix} x(t) + \tau\dot{x}(t) \\ \dot{x}(t) \end{pmatrix}$$

and covariance

$$q'\tau \begin{pmatrix} \dfrac{\tau^2}{3} & \dfrac{\tau}{2} \\ \dfrac{\tau}{2} & 1 \end{pmatrix} .$$

4 THE DISTRIBUTION OF CHANGE IN TARGET HEADING AND A METHOD FOR SELECTING THE MANOEUVRE PARAMETER

4.1 Discrete case

In this section, the pdf of the change in heading, over a given period of time, of a target obeying the second-order model is given. The details of the derivation of this pdf are given in Appendix B. The distribution of heading change is the basis of a method

for choosing a suitable manoeuvre parameter q for a given class of targets. The information required for choosing q is an estimate of the mean speed v_0 of the targets and the likely maximum change of target heading in some time period.

First consider a target moving in two dimensions with each Cartesian co-ordinate of the target governed by the second-order target model (equation (8)) with manoeuvre parameter q. The change in the heading angle of the target could be positive or negative, but since it is the magnitude of the change which is of interest, only the modulus of the change in heading angle is considered. In Appendix B the pdf of the change in heading angle θ after a period $\ell\Delta t$ (where θ takes values in the range $[0,\pi)$) has been derived:

$$p(\theta) = \frac{e^{-\frac{1}{2\zeta\ell}}}{\pi}\left\{1 + \sqrt{\pi}\xi e^{\xi^2}\left(1 + \operatorname{erf}(\xi)\right)\right\} , \qquad (13)$$

where $\xi = \dfrac{\cos\theta}{\sqrt{2\zeta\ell}}$,

$$\zeta = q\left(\frac{\Delta t}{v_0}\right)^2$$

and $\operatorname{erf}(.)$ is the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt .$$

The distribution function of θ defined by

$$F(\theta) \triangleq \int_{\lambda=0}^{\theta} p(\lambda)d\lambda$$

is the probability of the change in heading angle being less than θ, and does not appear to have any simple analytical representation. $F(\theta)$ may be interpreted as the probability that the target vector remains within a sector of semi-angle θ with axis of symmetry along the original target heading (see Fig 8a). It can be seen that $p(\theta)$ and $F(\theta)$ *depend only on the parameter* $\zeta\ell$ *and are independent of the initial target heading.* Figs 9 and 10 show $p(\theta)$ and $F(\theta)$ for $\zeta\ell$ taking the values 10^{-2}, 10^{-1}, 1, 10, 10^2 and 10^3. From equations (13), as $\zeta\ell \to \infty$ so $p(\theta) \to 1/\pi$ and $F(\theta) \to \theta/\pi$ (also see Figs 9 and 10), indicating that for large values of ζ or after a large number of time steps *the distribution of* θ *becomes uniform* (*ie* there is no preferential target heading) - as would be expected.

To make use of the distribution of θ for the selection of a suitable value of q for modelling a class of targets, contours of constant value of $F(\theta)$ have been plotted in Fig 11 as a function of θ and the distribution parameter $\zeta\ell$. Contours are shown for $F(\theta)$ taking the values of 0.3, 0.6, 0.9, 0.95 and 0.99, for the distribution parameter $\zeta\ell$ in the range $[10^{-3}, 10^2]$. If, for the class of targets of interest, the value

of θ for which $F(\theta)$ was equal to one of the contour values were known, then the required value of the distribution parameter $\zeta\ell$ (which determines q) could be directly read from Fig 11. However such precise information is unlikely to be available, although there is good chance of the tracking filter designer having some estimate of the likely maximum change of heading within some time period. If this likely maximum change of heading is formally defined as the value of θ for which there is a 95% probability of the target's change of heading being less than θ, then this value may be associated with the $F(\theta) = 0.95$ contour of Fig 11. Hence if, for the class of targets of interest, the maximum likely heading change in a time $\ell\Delta t$ is approximately θ', then a suitable value of q for modelling these targets is given by (see definition of ζ with equation (13))

$$q = \left\{\frac{(\zeta\ell)'}{\ell}\right\}\left(\frac{v_0}{\Delta t}\right)^2 ,$$

where $(\zeta\ell)'$ is read from Fig 11 as the value corresponding to $F(\theta')$ in the region of 0.95. For example consider a class of targets with a mean speed of about 15 m/s with an expected maximum heading change of $\pi/4$ in 13 seconds (corresponding to a turning circle of radius 250 metres). From Fig 11, $F(\pi/4) = 0.95$ corresponds to a value of $\zeta\ell = 0.15$, which indicates that a suitable value of q would be about (for $\Delta t = 1$ second),

$$q = \frac{0.15}{13}\left(\frac{v_0}{\Delta t}\right)^2 \simeq 2.6 \quad (m/s^2)^2 .$$

To illustrate the type of trajectories generated by different manoeuvre parameters, Figs 15 to 19 show realizations of target trajectories for $q = \zeta(v_0/\Delta t)^2$ where ζ takes the values 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2} and 10^{-1}. Each Figure shows six realizations of a 100 time step trajectory (solid line), as well as the expected trajectory (dotted line) and the 95% likelihood ellipse for the last point in the trajectory (given the exact initial conditions). When considering the change in heading angle over the hundred time steps ($\ell = 100$), the parameter $\zeta\ell$ takes the values 10^{-3}, 10^{-2}, 10^{-1}, 1 and 10 for Figs 15 to 19 respectively. For the admittedly limited sample, the observed change of heading is in agreement with the theoretical distribution (Fig 11). For example in Fig 17 for which $\zeta\ell = 0.1$, only one of the six trajectory realisations shows a heading change as much as 0.5 radian ($F(0.5) = 0.9$) and for the remaining examples the change is less than 0.25 ($F(0.25) = 0.6$).

The pdf of the change in heading angle after ℓ time steps has also been derived for a target moving in three dimensions (see Appendix B):

$$p(\theta) = \frac{e^{-\frac{1}{2\zeta\ell}}}{\sqrt{\pi}}\left\{\xi + \frac{\sqrt{\pi}}{2}e^{\xi^2}(1 + 2\xi^2)(1 + \mathrm{erf}(\xi))\right\}\sin\theta , \qquad (14)$$

where $\xi = \dfrac{\cos\theta}{\sqrt{2\zeta\ell}}$

and $\zeta = q\left(\dfrac{\Delta t}{v_0}\right)^2$.

Also, unlike the two-dimensional case, the distribution function may be simply expressed (see Appendix B):

$$F(\theta) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{1}{\sqrt{2\zeta\ell}}\right)\right) - \frac{e^{-\frac{1}{2\zeta\ell}}}{2} e^{\xi^2}\left(1 + \mathrm{erf}(\xi)\right) \cos\theta \quad . \tag{15}$$

In three dimensions, $F(\theta)$ may be interpreted as the probability that the target velocity vector remains inside a cone (of infinite extent) of semi-angle θ, with apex at the origin and axis of symmetry along the original target heading (see Fig 8b). As for the two-dimensional case, $p(\theta)$ and $F(\theta)$ are independent of the initial target heading and depend only on the distribution parameter $\zeta\ell$ ($p(\theta)$ and $F(\theta)$ are shown in Figs 12 and 13 for $\zeta\ell$ taking values of 10^{-2}, 10^{-1}, 1, 10, 10^2 and 10^3). As $\zeta\ell \to \infty$, equations (14) and (15) indicate that

$$p(\theta) \to \frac{\sin\theta}{2}$$

and

$$F(\theta) \to \sin^2\left(\frac{\theta}{2}\right) . \tag{16}$$

Since the proportion of the total volume of space enclosed by the infinite cone of semi-angle θ increases as $\sin^2(\theta/2)$, the distribution function (16) corresponds to a velocity vector pdf which has spherical symmetry about the origin (see Appendix B for a more rigorous argument). Hence, as would be expected, as $\zeta\ell \to \infty$ there is no preferential target heading. Fig 14 shows contours of $F(\theta)$ taking the values of 0.3, 0.6, 0.9, 0.95 and 0.99 for $\zeta\ell$ in the range $[10^{-3}, 10^2]$. This Figure can be used for choosing a suitable value of q for modelling targets moving in three dimensions in the same way as for the two-dimensional case.

The following points concerning this technique for selecting q should be considered:

(i) The technique can only be sensibly employed for expected maximum heading angle changes in the range $[0.1, 2.8]$ radians, which corresponds approximately to $\zeta\ell$ varying between 10^{-3} and 1. Outside this range, a small change in maximum heading angle results in very large changes in the parameter $\zeta\ell$ and since expected maximum heading change is unlikely to be known very precisely this could lead to large errors in the estimate of q.

(ii) For the class of targets of interest, an estimate of the maximum likely heading change in ℓ time steps will probably be available for several different values of ℓ, and for each of these a corresponding value for q can be obtained. If the values of q do not differ by more than one order of magnitude, then it is suggested that the average value of q should be taken. However, if the values of q vary over several orders of magnitude, the second order model may be an inadequate representation of the class of targets (*ie* the random walk model for target manoeuvre is incorrect) and it may be necessary to consider an alternative model.

(iii) In selecting the value of q on the basis of the maximum likely change in heading (and using the second-order model), there is an implicit assumption that the distribution of heading angle change for the class of targets of interest is similar to the corresponding $F(\theta)$.

(iv) The value of q could be determined via some other statistic of the change in heading angle if this information were available rather than an estimate of the likely maximum change. For example, for certain applications the mean change of heading angle $\bar{\theta}$ after some period might be available. To determine q from this information, the relationship between $\bar{\theta}$ and $\zeta\ell$ would have to be obtained via numerical integration using equations (13) or (14).

(v) This technique for selecting q can only be expected to give an order of magnitude estimate. However, this is probably adequate for many applications since the performance of tracking filters is sensitive to the order of magnitude of q rather than to the exact value, and also the lack of information (or uncertainty) in the likely manoeuvres for a target class will often preclude a more precise determination of q . If further information on the target trajectories likely to be encountered is available, it may be possible to make a more precise determination of q by simulation.

4.2 Continuous case

The equivalent manoeuvre parameter for the continuous second-order model is q' , which is the constant power spectral density of the acceleration noise (see section 2). The pdf of change of target heading in τ seconds for the continuous case is also given by equations (13) and (14) for the two-dimensional and three-dimensional cases respectively, if the distribution parameter $\zeta\ell$ is replaced by $\eta\tau$, where η is given by $\eta = q'/v_0^2$ and has units of s^{-2}/Hz (if the power spectral density q' is with respect to Hz). Hence Figs 11 and 14 may also be used for choosing the manoeuvre parameter for the continuous model. If the value of $(\zeta\ell)$ appropriate to the known likely maximum heading change in time τ is read from Figs 11 or 14, as indicated in section 4.1, then a suitable value of q' for the continuous model is given by

$$q' = \frac{(\zeta\ell)}{\tau} v_0^2 .$$

5 CONCLUSIONS

The statistics of the trajectories generated by a second-order target model have been presented. The mean or expected target trajectory is a constant velocity path. The uncertainty (*ie* the variance) in the future position of the target about the expected path, given perfect knowledge of the current position and velocity, increases approximately as the cube of time. The variance of the target velocity increases linearly with time. Both the position and velocity variance are also proportional to the acceleration noise variance q in the discrete formulation, or the constant power spectral density q' in the continuous case.

Using the statistics of the target trajectories, the probability density function of the change in target heading over a given time period has been derived. This distribution

forms the basis of a new technique for choosing the manoeuvre parameter q which is suitable for a given class of targets. Design curves are supplied which can be used for selecting q, given knowledge of the likely maximum change of target heading in some period and the mean target speed.

Appendix A

THE MARKOV PROPERTY AND LINEAR SYSTEMS DRIVEN BY GAUSSIAN NOISE

A.1 Discrete systems

Consider a sequence of random vectors:

$$\{\underline{x}_k\}_{k=0}^{N} \quad .$$

This random sequence is completely determined statistically if the joint probability density function (pdf) of the elements of the sequence is known:

$$p(\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_N) \quad . \tag{A-1}$$

A random sequence is said to possess the Markov property if for all k,

$$p(\underline{x}_{k+1} | \underline{x}_k, \underline{x}_{k-1}, \ldots, \underline{x}_0) = p(\underline{x}_{k+1} | \underline{x}_k)$$

so that the pdf of $\underline{x}_{k+1}$ depends only on $\underline{x}_k$. That is, if the subscript k corresponds to increasing time, for a Markov sequence, the future probabilistic behaviour of the sequence depends only on a knowledge of the present. It can be shown (see Bryson and Ho[5]) that for a Markov sequence the joint pdf (A-1) is completely described by the initial pdf $p(\underline{x}_0)$ and the transition density function $p(\underline{x}_{k+1} | \underline{x}_k)$. A Markov sequence is said to be homogeneous if the transition density function is independent of k. A Markov sequence is stationary if all the random variables $\underline{x}_k$ have the same pdf.

A Gauss-Markov random sequence is a Markov sequence for which $p(\underline{x}_k)$ and $p(\underline{x}_{k+1} | \underline{x}_k)$ are Gaussian pdf. The sequence of state vectors resulting from a linear system driven by Gaussian white noise is a Gauss-Markov sequence (see Bryson and Ho[5]), *ie* if

$$\underline{x}_{k+1} = \phi_k \underline{x}_k + \Gamma \underline{w}_k \quad ,$$

where $\underline{x}_k$ is the state vector,
ϕ_k is the state transition matrix from k to $k+1$
and Γ is a matrix
and $\underline{w}_k$ is a random Gaussian sequence with $E\left[\underline{w}_k\right] = 0$ *and* $E\left[\underline{w}_k \underline{w}_j^T\right] = Q\delta_{kj}$,

then the sequence $\{\underline{x}_k\}$ resulting from this system is a Gauss-Markov sequence with

$$p(\underline{x}_{k+1} | \underline{x}_k) = \mathscr{N}\left(\underline{x}_{k+1} ; \phi_k \underline{x}_k, \Gamma Q \Gamma^T\right) \quad , \tag{A-2}$$

where $\mathscr{N}(\underline{x};\underline{\mu},V)$ represents the Gaussian pdf with mean $\underline{\mu} = E[\underline{x}]$ and covariance $V = E[(\underline{x} - \underline{\mu})(\underline{x} - \underline{\mu})^T]$. Using the Chapman-Kolmogorov equation:

$$p(\underline{x}_{k+\ell} | \underline{x}_k) = \int_{-\infty}^{\infty} p(\underline{x}_{k+\ell} | \underline{x}_{k+m}) p(\underline{x}_{k+m} | \underline{x}_k) d\underline{x}_{k+m} \quad ,$$

where $0 < m < \ell$, and (A-2), it may be shown that

$$p(\underline{x}_{k+\ell}|\underline{x}_k) = \mathcal{N}(\underline{x}_{k+\ell};\underline{\mu},C) \quad ,$$

where

$$\underline{\mu} = \Phi(k+\ell,k)\underline{x}_k \tag{A-3}$$

and

$$C = \sum_{n=k}^{k+\ell-1} \Phi(k+\ell,n+1)\Gamma Q\Gamma^T\Phi^T(k+\ell,n+1) \tag{A-4}$$

with

$$\Phi(k+\ell,k) = \phi_{k+\ell-1}\phi_{k+\ell-2}\cdots\phi_k \quad .$$

If the state transition matrix ϕ_k is independent of k (*ie* $\phi_k = \phi$ for all k), then the Gauss-Markov sequence $\{\underline{x}_k\}$ is homogeneous and (A-3) and (A-4) reduce to

$$\underline{\mu} = \phi^{\ell}\underline{x}_k \quad , \tag{A-5}$$

$$C = \sum_{n=0}^{\ell-1} \phi^{\ell-n-1}\Gamma Q\Gamma^T{\phi^{\ell-n-1}}^T \quad . \tag{A-6}$$

In this case it can be seen that $p(\underline{x}_{k+\ell}|\underline{x}_k)$ is independent of k .

Suppose that the pdf of the state vector $\underline{x}_k$ is given by

$$p(\underline{x}_k) = \mathcal{N}(\underline{x}_k;\bar{\underline{x}}_k,M_k)$$

and that $\bar{\underline{x}}_k$ and M_k are known but $\underline{x}_k$ is unknown.
In this case it can be shown that

$$p(\underline{x}_{k+\ell}|\bar{\underline{x}}_k,M_k) = \mathcal{N}(\underline{x}_{k+\ell};\bar{\underline{x}}_{k+\ell},M_{k+\ell}) \quad ,$$

where

$$\bar{\underline{x}}_{k+\ell} = \Phi(k+\ell,k)\bar{\underline{x}}_k \tag{A-7}$$

and

$$M_{k+\ell} = \Phi(k+\ell,k)M_k\Phi^T(k+\ell,k) + \sum_{n=k}^{k+\ell-1} \Phi(k+\ell,n+1)\Gamma Q\Gamma^T\Phi^T(k+\ell,n+1) \quad . \tag{A-8}$$

If $\phi_k = \phi$ for all k , then

$$\bar{\underline{x}}_{k+\ell} = \phi^{\ell}\bar{\underline{x}}_k \tag{A-9}$$

and

$$M_{k+\ell} = \phi^{\ell} M_k \phi^{\ell^T} + \sum_{n=0}^{\ell-1} \phi^{\ell-n-1} \Gamma Q \Gamma^T \phi^{\ell-n-1^T} \quad . \tag{A-10}$$

The above equations describe the propagation of the mean and the covariance of the sequence. Note that, as would be expected, (from (A-3), (A-4), (A-7) and (A-8)) when $M_k = 0$, indicating that $\bar{\underline{x}}_k = \underline{x}_k$,

$$p(\underline{x}_{k+\ell} | \bar{\underline{x}}_k, M_k) = p(\underline{x}_{k+\ell} | \underline{x}_k) \quad .$$

Finally the correlation matrix between members of the sequence is given by

$$R_{k+\ell\ k} \triangleq E\left[(\underline{x}_{k+\ell} - \bar{\underline{x}}_{k+\ell})(\underline{x}_k - \bar{\underline{x}}_k)^T\right]$$

$$= \Phi(k+\ell,k)M_k$$

and

$$R_{k\ k+\ell} = M_k \Phi^T(k+\ell,k) \quad .$$

A.2 Continuous systems

The definitions and relations for continuous systems are analogous to those for discrete systems. A continuous random (or stochastic) process $\underline{x}(t)$ is Markov if, for any $t_1 < t_2 < \ldots < t_n$,

$$p\left(\underline{x}(t_n) | \underline{x}(t_{n-1}), \underline{x}(t_{n-2}), \ldots, \underline{x}(t_1)\right) = p\left(\underline{x}(t_n) | \underline{x}(t_{n-1})\right) \quad .$$

A Markov process is completely determined statistically in the interval (t_0, t_f) if the density functions $p\left(\underline{x}(t_1) | \underline{x}(t_2)\right)$ and $p(\underline{x}(t_0))$ are known for all $t_1 < t_2$ in the interval (t_0, t_f). A Markov process is homogeneous if $p\left(\underline{x}(t+\tau) | \underline{x}(t)\right)$ is independent of t.

A Gauss-Markov process is a Markov process for which $p\left(\underline{x}(t) | \underline{x}(\tau)\right)$ and $p\left(\underline{x}(t)\right)$ are Gaussian pdf. The state vector of a continuous linear system driven by Gaussian white noise is a Gauss-Markov process, *ie* if

$$\dot{\underline{x}} = F\underline{x} + G\underline{w}' \quad , \tag{A-11}$$

where $\underline{x}$ is the state vector, F and G are matrices and $\underline{w}'$ is a vector of white Gaussian noise such that

$$E[\underline{w}'] = 0$$

and

$$E[\underline{w}'(t+\tau)\underline{w}'^T(t)] = Q'\delta(\tau) \quad ,$$

then the output $\underline{x}(t)$ of this system is a Gauss-Markov process.

Expressions which are analogous to the discrete case, may be derived for the conditional pdf of output $\underline{x}(t)$ of continuous system (A-11) (see Bryson and Ho[5]):

$$p\left(\underline{x}(t+\tau)\,|\,\underline{x}(t)\right) = \mathcal{N}\left(\underline{x}(t+\tau);\underline{\mu}',C'\right) \quad ,$$

where

$$\underline{\mu}' = \Phi'(t+\tau,t)\underline{x}(t) \tag{A-12}$$

and

$$C' = \int_t^{t+\tau} \Phi'(t+\tau,\alpha)GQ'G^T\Phi'^T(t+\tau,\alpha)d\alpha \quad , \tag{A-13}$$

where $\Phi'(t_1,t_2)$ is the transition matrix of (A-11).

If $\Phi'(t+\tau,t)$ is only dependent on τ , *ie*

$$\Phi'(t+\tau,t) = \phi'(\tau) \quad \text{for all } t \;,$$

then (A-12) and (A-13) reduce to

$$\underline{\mu}' = \phi'(\tau)\underline{x}(t) \tag{A-14}$$

and

$$C' = \int_0^{\tau} \phi'(\tau-\alpha)GQ'G^T\phi'^T(\tau-\alpha)d\alpha \quad . \tag{A-15}$$

In this case it can be seen that $p\left(\underline{x}(t+\tau)\,|\,\underline{x}(t)\right)$ is independent of t and so the Gauss-Markov process is homogeneous.

If the pdf of $\underline{x}(t)$ is known to be

$$p(\underline{x}(t)) = \mathcal{N}\left(\underline{x}(t);\bar{\underline{x}}(t),M(t)\right)$$

then

$$p\left(\underline{x}(t+\tau)\,|\,\bar{\underline{x}}(t),M(t)\right) = \mathcal{N}\left(\underline{x}(t+\tau);\bar{\underline{x}}(t+\tau),M(t+\tau)\right) \quad ,$$

where

$$\bar{\underline{x}}(t+\tau) = \Phi'(t+\tau,t)\bar{\underline{x}}(t) \tag{A-16}$$

and

$$M(t+\tau) = \Phi'(t+\tau,t)M(t)\Phi'^T(t+\tau,t) + \int_t^{t+\tau} \Phi'(t+\tau,\alpha)GQ'G^T\Phi'^T(t+\tau,\alpha)d\alpha \tag{A-17}$$

which is the solution of

$$\frac{dM}{dt} = FM + MF^T + GQ'G^T \quad .$$

Finally the correlation matrix of the process is given by

$$R(t+\tau,t) = \Phi'(t+\tau,t)M(t)$$

and

$$R(t,t+\tau) = M(t)\Phi'^{T}(t+\tau,t) \quad .$$

Appendix B

DERIVATION OF THE DISTRIBUTION OF CHANGE IN TARGET HEADING

B.1 Two-dimensional case

In this Appendix, the pdf of the change in heading, over a given period of time, for a target obeying the second-order model (equation (8) of section 2) is derived. Suppose that a target moving in the x-y plane has a velocity $\begin{pmatrix}\dot{x}_0\\ \dot{y}_0\end{pmatrix}$ at some instant. The velocity $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ at a later time has a pdf which may be determined from the results of section 3. If each co-ordinate is independent and each has the same manoeuvre parameter q, then, for the discrete target model, equation (12) of section 3 indicates that each velocity component has a Gaussian distribution with variance $q(\Delta t)^2 \ell$ and mean $\dot{x}_0$ or $\dot{y}_0$ after a time interval of $\ell\Delta t$. Hence

$$p(\dot{x},\dot{y}|\dot{x}_0,\dot{y}_0) = \frac{1}{2\pi a^2} \exp\left[-\frac{1}{2a^2}\left\{(\dot{x} - \dot{x}_0)^2 + (\dot{y} - \dot{y}_0)^2\right\}\right] \quad , \qquad \text{(B-1)}$$

where $a^2 = q(\Delta t)^2 \ell$.

The change of heading λ, is the angle between the vectors $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ and $\begin{pmatrix}\dot{x}_0\\ \dot{y}_0\end{pmatrix}$. Since only the absolute value of the change of heading is of interest, define the probability distribution function $F(\theta)$ as the probability that λ is within the range $[-\theta,\theta]$. Hence $F(\theta)$ is given by

$$F(\theta) = \iint_D p(\dot{x},\dot{y}|\dot{x}_0,\dot{y}_0)\,d\dot{x}\,d\dot{y} \quad , \qquad \text{(B-2)}$$

where D is the region of the $(\dot{x},\dot{y})$ plane for which $\lambda \epsilon [-\theta,\theta]$.

To write the integral (B-2) in a form more amenable to analysis, consider a change of variables from $(\dot{x},\dot{y})$ to (r,λ) where $r^2 = \dot{x}^2 + \dot{y}^2$ and λ is as defined above. Using the cosine rule (see Fig B1a), the argument of the exponential in (B-1) may be written

$$(\dot{x} - \dot{x}_0)^2 + (\dot{y} - \dot{y}_0)^2 = r^2 + r_0^2 - 2rr_0 \cos\lambda \quad ,$$

where $r_0 = \sqrt{\dot{x}_0^2 + \dot{y}_0^2}$.

Hence (B-2) becomes

$$F(\theta) = \frac{1}{2\pi a^2} \int_{\lambda=-\theta}^{\theta} \int_{r=0}^{\infty} \exp\left[-\frac{1}{2a^2}\left(r^2 + r_0^2 - 2rr_0\cos\lambda\right)\right] r\,dr\,d\lambda \quad . \qquad \text{(B-3)}$$

Since the integrand is symmetric with respect to λ, changing the range of integration over λ to $[0,\theta]$ and doubling the value of the right-hand side of (B-3) leaves the value of $F(\theta)$ unchanged. If also an integration by parts on the integral over r is performed, then (B-3) becomes

$$F(\theta) = \frac{1}{\pi}\int_{\lambda=0}^{\theta}\left\{e^{-r_0^2/2a^2} + \frac{r_0 \cos\lambda}{a^2}\int_{r=0}^{\infty}\exp\left[-\frac{1}{2a^2}\left(r^2 + r_0^2 - 2rr_0\cos\lambda\right)\right]dr\right\}d\lambda .$$

The remaining integral over r can be evaluated (see Abramowitz and Stegun[6], 7.4.2) and so

$$F(\theta) = \frac{e^{-r_0^2/2a^2}}{\pi}\int_{\lambda=0}^{\theta} 1 + \sqrt{\pi}\xi e^{\xi^2}(1 + \mathrm{erf}(\xi))d\lambda \quad , \tag{B-4}$$

where $\xi = \dfrac{r_0 \cos\lambda}{\sqrt{2}\, a}$.

Now if the initial speed of the target was v_0 , so that $v_0 = r_0$, and if q is written as

$$q = \zeta\left(\frac{v_0}{\Delta t}\right)^2 \quad ,$$

where ζ is a non-dimensional parameter, then (B-4) may be written

$$F(\theta) = \frac{e^{-\frac{1}{2\zeta\ell}}}{\pi}\int_{\lambda=0}^{\theta} 1 + \sqrt{\pi}\,\xi e^{\xi^2}(1 + \mathrm{erf}(\xi))d\lambda \quad , \tag{B-5}$$

where $\xi = \dfrac{\cos\lambda}{\sqrt{2\zeta\ell}}$.

Hence the distribution function of heading change depends only on the parameter $\zeta\ell$. Using numerical integration $F(\theta)$ has been plotted for different values of $\zeta\ell$ in Fig 10. The pdf of heading change is given by

$$p(\theta) = \frac{dF(\theta)}{d\theta}$$

and so directly from (B-5)

$$p(\theta) = \frac{e^{-\frac{1}{2\zeta\ell}}}{\pi}\left\{1 + \sqrt{\pi}\,\xi\, e^{\xi^2}(1 + \mathrm{erf}(\xi))\right\} \quad , \tag{B-6}$$

where $\xi = \dfrac{\cos\theta}{\sqrt{2\zeta\ell}}$.

$p(\theta)$ is shown in Fig 9 for different values of the parameter $\zeta\ell$.

B.2 Three-dimensional case

By a simple extension of (B-1), the conditional pdf of velocity for the three-dimensional case is given by

$$p(\dot{x},\dot{y},\dot{z}|\dot{x}_0,\dot{y}_0,\dot{z}_0) = \frac{1}{(2\pi)^{\frac{3}{2}}a^3}\exp\left\{-\frac{1}{2a^2}\left[(\dot{x}-\dot{x}\right.\right.$$ [illegible]

The probability distribution function $F(\theta)$ for this case is [illegible] that the target velocity vector remains within a cone [illegible] angle θ, with apex at the origin and axis of symmetry along the [illegible] $F(\theta)$ can therefore be derived by integrating (B-7) over the [illegible] facilitate this consider a spherical co-ordinate system with elevation [illegible] from the vector $\begin{pmatrix}\dot{x}_0\\ \dot{y}_0\\ \dot{z}_0\end{pmatrix}$, azimuthal angle ψ measured with respect to some fixed plane [illegible] contains the vector $\begin{pmatrix}\dot{x}_0\\ \dot{y}_0\\ \dot{z}_0\end{pmatrix}$ and radial distance r given by

$$r^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 .$$

From the cosine rule, the argument of the exponential in (B-7) can be written (see Fig B1b)

$$(\dot{x}-\dot{x}_0)^2 + (\dot{y}-\dot{y}_0)^2 + (\dot{z}-\dot{z}_0)^2 = r^2 + r_0^2 - 2rr_0\cos\lambda ,$$

where $r_0 = \sqrt{\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2}$.

Also for a spherical co-ordinate system, the elemental volume is

$$r^2 \sin\lambda \, dr d\psi d\lambda .$$

Hence the distribution function of heading change is given by

$$F(\theta) = \frac{1}{(2\pi)^{\frac{3}{2}}a^3}\int_{r=0}^{\infty}\int_{\lambda=0}^{\theta}\exp\left\{-\frac{1}{2a^2}\left(r^2 + r_0^2 - 2rr_0\cos\lambda\right)\right\} r^2\sin\lambda \, d\lambda dr \int_{\psi=0}^{2\pi} d\psi . \quad \text{(B-8)}$$

The integration with respect to λ can be performed by inspection, and with evaluating the integral over ψ, this gives

$$F(\theta) = \frac{1}{\sqrt{2\pi}a}\int_{r=0}^{\infty}\frac{r}{r_0}\exp\left\{-\frac{\left(r^2 + r_0^2\right)}{2a^2}\right\}\left(e^{rr_0/a^2} - e^{rr_0\cos\theta/a^2}\right)dr . \quad \text{(B-9)}$$

(B-9) can be written as the sum of two integrals of the same form as the integral over r in (B-3) and hence can be integrated in the same manner to give

$$F(\theta) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{r_0}{\sqrt{2}a}\right)\right) - \frac{1}{2}e^{-r_0^2/2a^2} e^{\xi^2}(1 + \mathrm{erf}(\xi)) \cos\theta \quad ,$$

with $\xi = \dfrac{r_0 \cos\lambda}{\sqrt{2}a}$. If the initial target speed was v_0 , so that $v_0 = r_0$, and if q is written as

$$q = \zeta\left(\frac{v_0}{\Delta t}\right)^2$$

then

$$F(\theta) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{1}{\sqrt{2\zeta\ell}}\right)\right) - \frac{e^{-\frac{1}{2\zeta\ell}}}{2} e^{\xi^2}(1 + \mathrm{erf}(\xi)) \cos\theta \quad , \tag{B-10}$$

where $\xi = \dfrac{\cos\theta}{\sqrt{2\zeta\ell}}$.

Hence the distribution function only depends on the parameter $\zeta\ell$, and in Fig 13 $F(\theta)$ is shown for different values of $\zeta\ell$. The density function $p(\theta)$ may be obtained by differentiating (B-10) to give

$$p(\theta) = \frac{e^{-\frac{1}{2\zeta\ell}}}{\sqrt{\pi}}\left\{\xi + \frac{\sqrt{\pi}}{2} e^{\xi^2}(1 + 2\xi^2)(1 + \mathrm{erf}(\xi))\right\} \sin\theta \quad , \tag{B-11}$$

where $\xi = \dfrac{\cos\theta}{\sqrt{2\zeta\ell}}$.

$p(\theta)$ is shown in Fig 12 for different values of $\zeta\ell$.

As $\zeta\ell \to \infty$, $\xi \to 0$ and

$$F(\theta) \to \sin^2\left(\frac{\theta}{2}\right) \quad ,$$

$$p(\theta) \to \frac{\sin\theta}{2} \quad .$$

This is the distribution of elevation angle which results from any pdf with spherical symmetry. To prove this consider any function f whose integral over all space is unity and which only depends on r . Hence, for any spherical co-ordinate system,

$$\int_{\psi=0}^{2\pi}\int_{\lambda=0}^{\pi}\int_{r=0}^{\infty} f(r)r^2 \sin\lambda \, dr\, d\lambda\, d\psi = 1$$

which implies

$$\int_{r=0}^{\infty} f(r)r^2 dr = \frac{1}{4\pi} \quad . \tag{B-12}$$

$$F(\theta) = \int_{\lambda=0}^{\theta} \sin\lambda \, d\lambda \int_{\psi=0}^{2\pi} d\psi \int_{r=0}^{\infty} f(r) r^2 dr$$

$$= (1 - \cos\theta) 2\pi \frac{1}{4\pi} = \sin^2\left(\frac{\theta}{2}\right)$$

using (B-12), which provides the required proof. So as would be expected, as $\zeta\ell \to \infty$ there is no preferential heading.

B.3 Alternative derivation

An alternative derivation of the distribution of change in heading angle using Student's non-central t distribution has been suggested by David Lloyd[11] of AW2 Division, RAE Farnborough. If t' is defined by

$$t' = \frac{x + \delta}{\chi / \sqrt{\nu}} , \qquad \text{(B-13)}$$

where $\chi = \sqrt{\sum_{k=1}^{\nu} x_k^2}$,

δ is some constant

and $x, x_1, x_2, \ldots, x_\nu$ are independent and Gaussian with zero mean and unit variance, then t' has the pdf

$$p(t'; \nu, \delta) = \frac{1}{\sqrt{\nu}\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi}\, 2^{\frac{\nu-1}{2}}} \exp\left[-\frac{\nu\delta^2}{2(t'^2 + \nu)}\right]\left(\frac{\nu}{\nu + t'^2}\right)^{\frac{\nu+1}{2}} \int_0^\infty w^\nu \exp\left[-\frac{1}{2}\left(w - \frac{\delta t'}{\sqrt{t'^2 + \nu}}\right)^2\right] dw ,$$

...... (B-14)

(which follows from (9.6.7) of Fisz[7]). (B-14) is an expression for the non-central t distribution.

First consider the three-dimensional case. Rotate the Cartesian co-ordinate system about the origin to form a new Cartesian system $(\dot{x}', \dot{y}', \dot{z}')$ with $\dot{z}'$ aligned along the initial target heading (the orientation of $\dot{x}'$ and $\dot{y}'$ is arbitrary from symmetry considerations). Then $\dot{x}'$, $\dot{y}'$ and $\dot{z}'$ are independent and Gaussian with variance a^2, and $\dot{x}'$ and $\dot{y}'$ have zero mean while $\dot{z}'$ has a mean v_0 (which is the initial target speed). The change in heading θ is the angle between the $\dot{z}'$ axis and the vector $(\dot{x}', \dot{y}', \dot{z}')^T$.

Hence

$$\cot\theta = \frac{\dot{z}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}}$$

$$= \frac{\dot{z}'' + (v_0/a)}{\sqrt{\dot{x}''^2 + \dot{y}''^2}} ,$$

where $\dot{x}''$, $\dot{y}''$ and $\dot{z}''$ are independent and Gaussian with zero mean and unit variance (see Ruben[8] for the general n dimensional case). Comparing this expression with (B-13), it can be seen that $\sqrt{2}\cot\theta$ has a non-central t distribution with $\nu = 2$ and $\delta = v_0/a$. Hence, the pdf of θ is given by

$$p(\theta) = \left[p\left(t';2,\frac{v_0}{a}\right)\left|\frac{dt'}{d\theta}\right|\right]_{t'=\sqrt{2}\cot\theta}$$

$$= \left[p\left(t';2,\frac{v_0}{a}\right)\right]_{t'=\sqrt{2}\cot\theta} \sqrt{2}\,\mathrm{cosec}^2\theta \quad . \tag{B-15}$$

Noting that $t' = \sqrt{2}\cot\theta$ implies that

$$\frac{2}{2+t'^2} = \sin^2\theta \qquad \text{and} \qquad \frac{t'}{\sqrt{t'^2+\nu}} = \cos\theta$$

and putting (B-14) into (B-15),

$$p(\theta) = \frac{\sin\theta}{\sqrt{2\pi}} \int_0^\infty w^2 \exp\left[-\frac{1}{2}\left(w^2 - 2\frac{v_0}{a} w\cos\theta + \frac{v_0^2}{a^2}\right)^2\right] dw$$

from which (B-8) of section B.2 follows using

$$F(\theta) = \int_0^\theta p(\lambda)d\lambda \quad . \tag{B-16}$$

Likewise for the two-dimensional case, the change in heading θ is given by

$$\cot\theta = \frac{\dot{y}'' + \frac{v_0}{a}}{\sqrt{\dot{x}''^2}} \quad ,$$

where $\dot{x}''$ and $\dot{y}''$ are Gaussian with zero mean and unit variance, so that $\cot\theta$ has a non-central t distribution with $\nu = 1$ and $\delta = v_0/a$. Hence

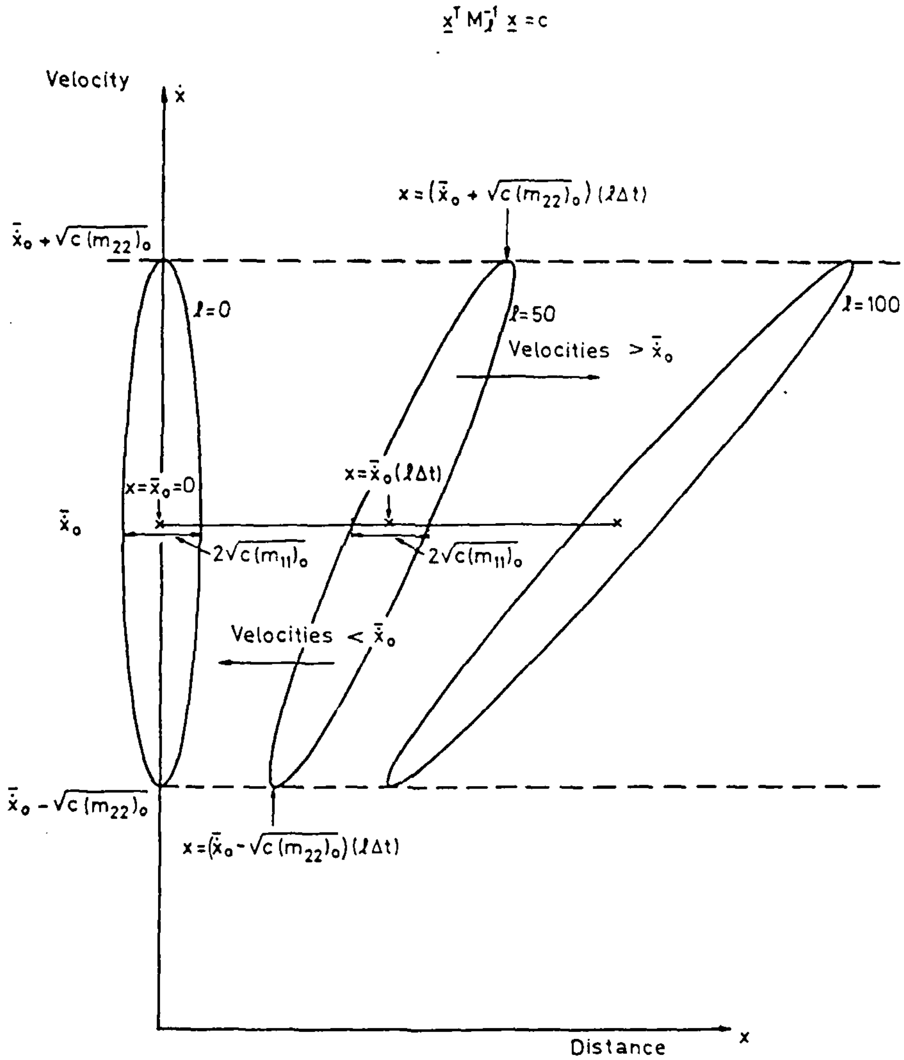
$$p(\theta) = \left[p\left(t';1,\frac{v_0}{a}\right)\right]_{t'=\cot\theta} \mathrm{cosec}^2\theta$$

$$= \frac{1}{\pi}\int_0^\infty w\exp\left[-\frac{1}{2}\left(w^2 - 2\frac{v_0}{a} w\cos\theta + \frac{v_0^2}{a^2}\right)^2\right] dw$$

from which (B-3) of section B.1 follows using (B-16).

REFERENCES

No.	Author	Title, etc
1	D.J. Salmond	The Kalman tracking filter, the α-β filter and smoothing filters. RAE Technical Memorandum AW 48 (1981)
2	A. Papoulis	Probability, random variables and stochastic processes. McGraw Hill (1981)
3	G. Jenkins D. Watts	Spectral analysis and its applications. Holden-Day (1968)
4	P. Sage J. Melsa	Estimation theory with applications to communications and control. McGraw Hill (1971)
5	A. Bryson Y-C. Ho	Applied optimal control. Halsted Press (1975)
6	M. Abramowitz I. Stegun	Handbook of mathematical functions. Dover (1965)
7	M. Fisz	Probability theory and mathematical statistics. John Wiley (1963)
8	H. Ruben	Probability content of regions under spherical normal distributions, I. Annals of Mathematical Statistics, Vol 31 (1960)
9	A.H. Jazwinski	Stochastic processes and filtering theory. Academic Press (1970)
10	A. Papoulis	Signal analysis. McGraw Hill (1977)
11	D.E. Lloyd	Private communication.



TR 85071

Fig 1 Evolving likelihood ellipses with no system driving noise

Fig 2

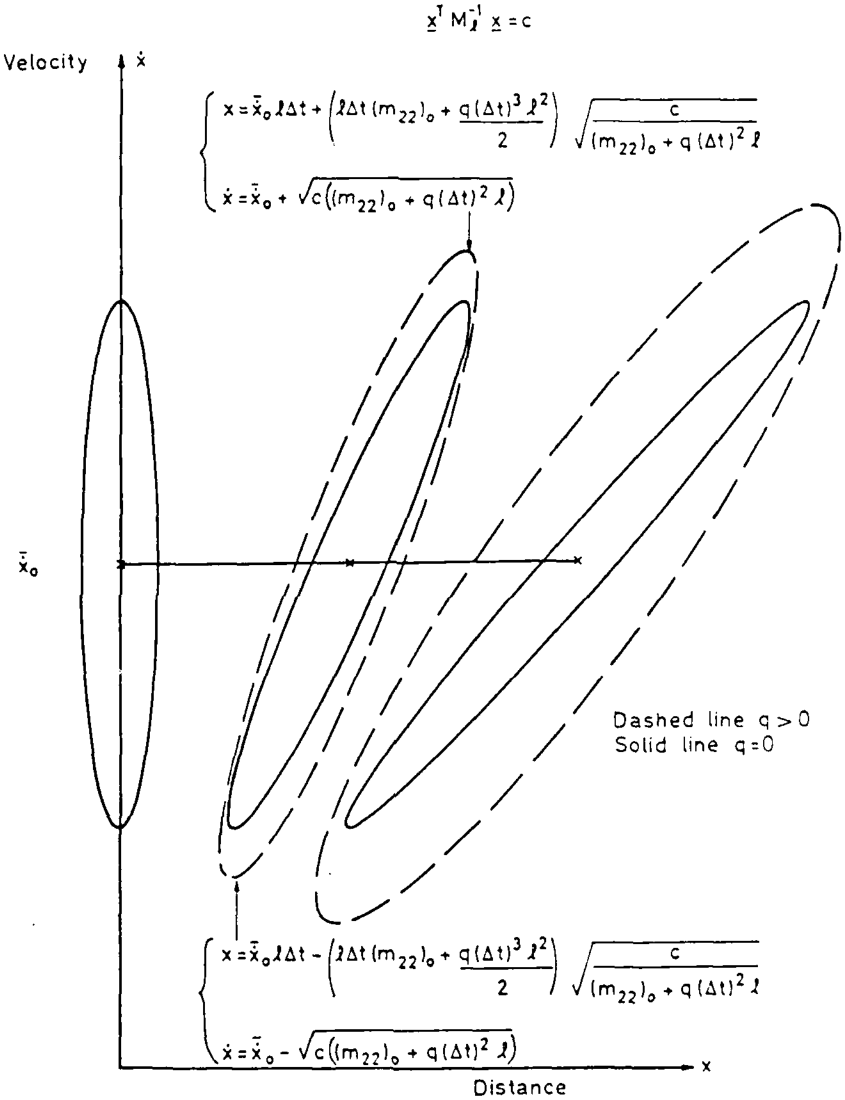


Fig 2 Likelihood ellipses for $q = 0$ and $q > 0$

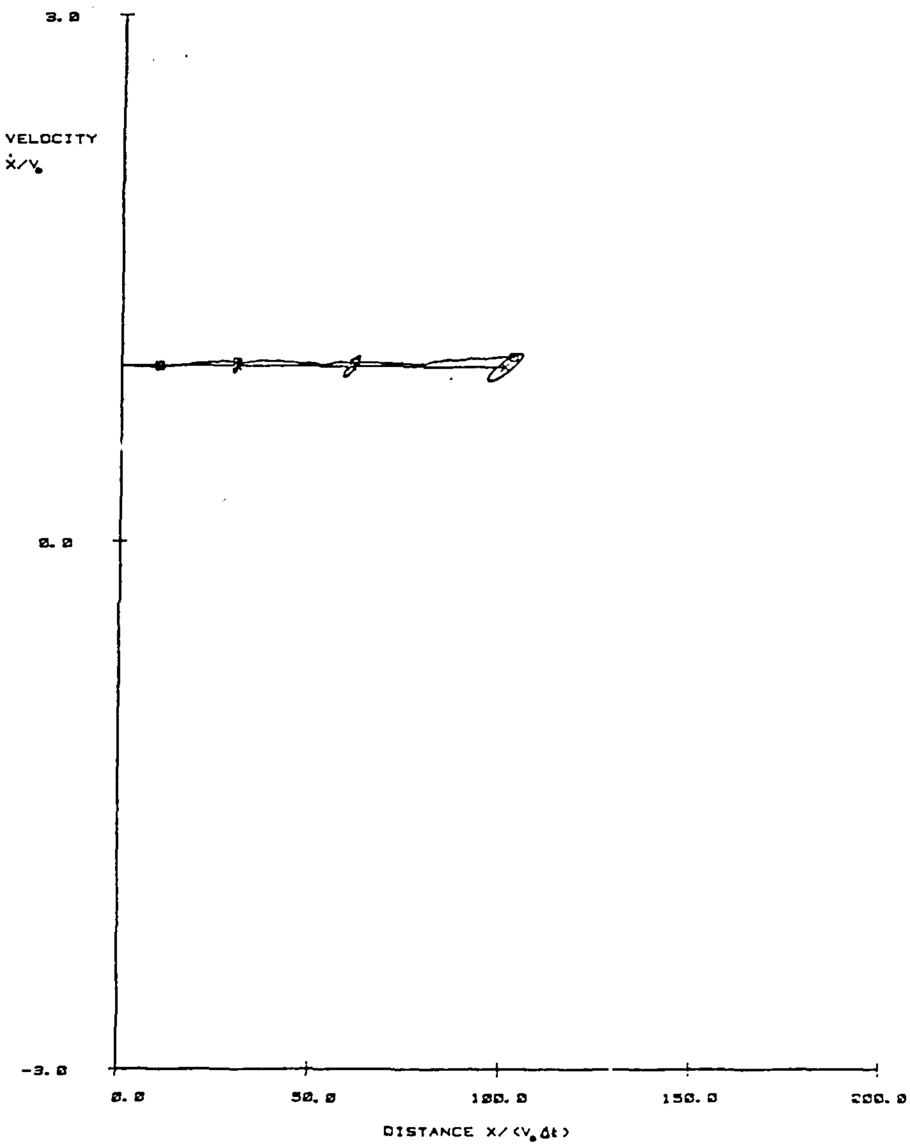


Fig 3 State space trajectory for $q = 10^{-5}\ (V_0/\Delta t)^2$

TR 85073

Fig 4

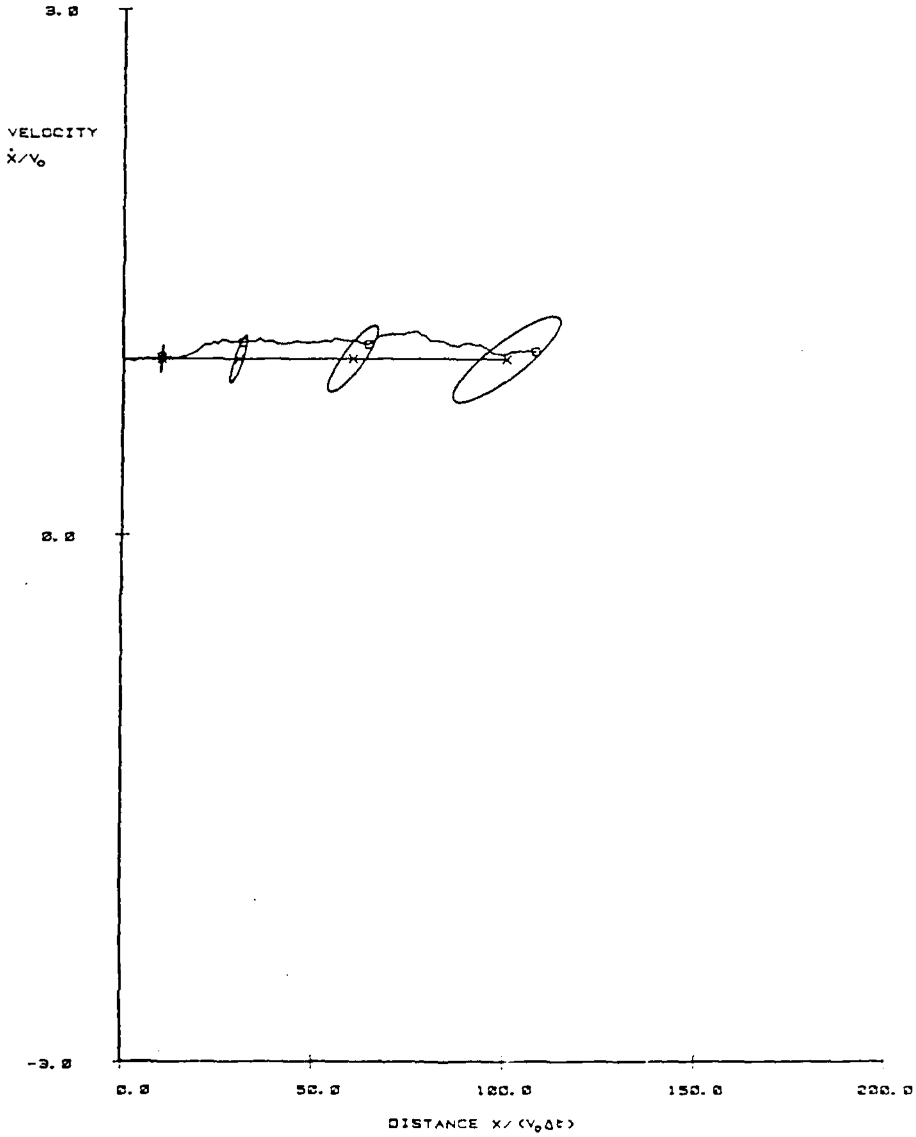


Fig 4 State space trajectory for $q = 10^{-4}\ (V_0/\Delta t)^2$

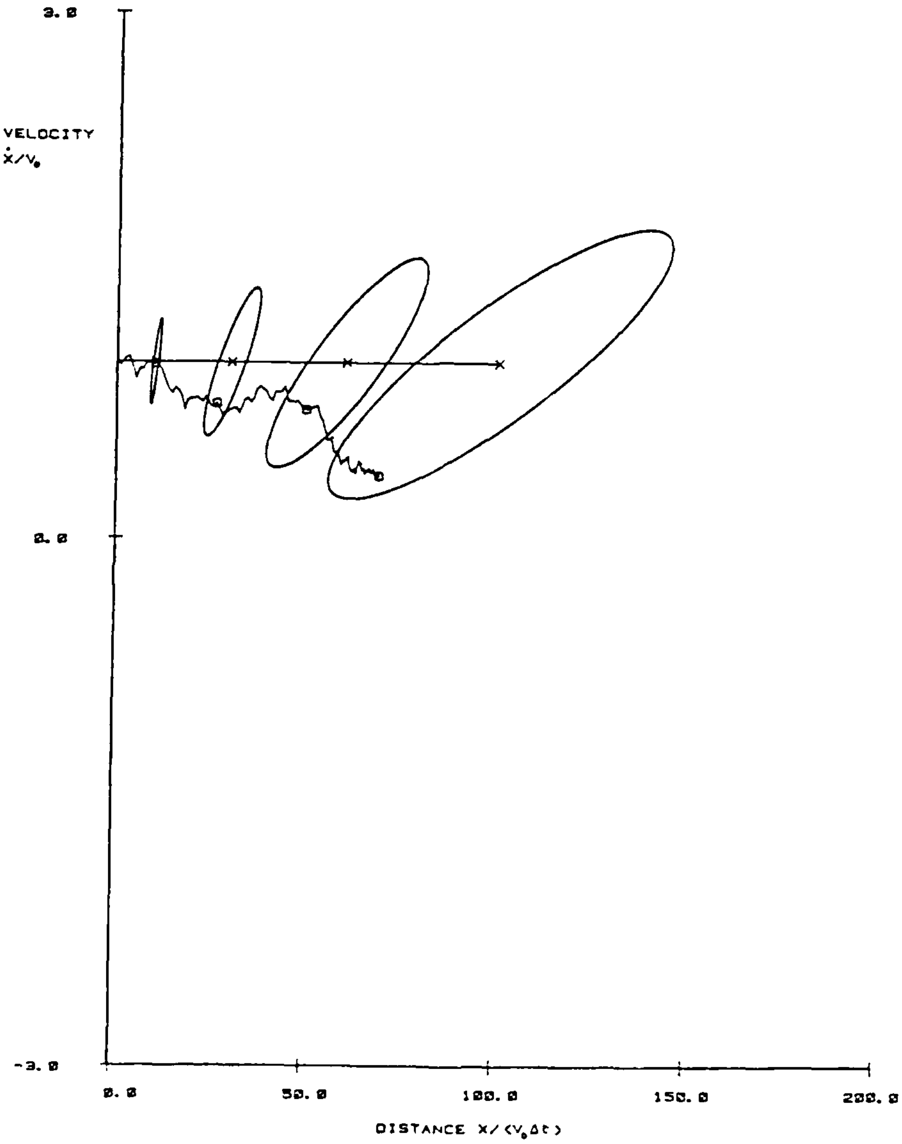


Fig 5 State space trajectory for $q = 10^{-3} (V_0/\Delta t)^2$

Fig 6

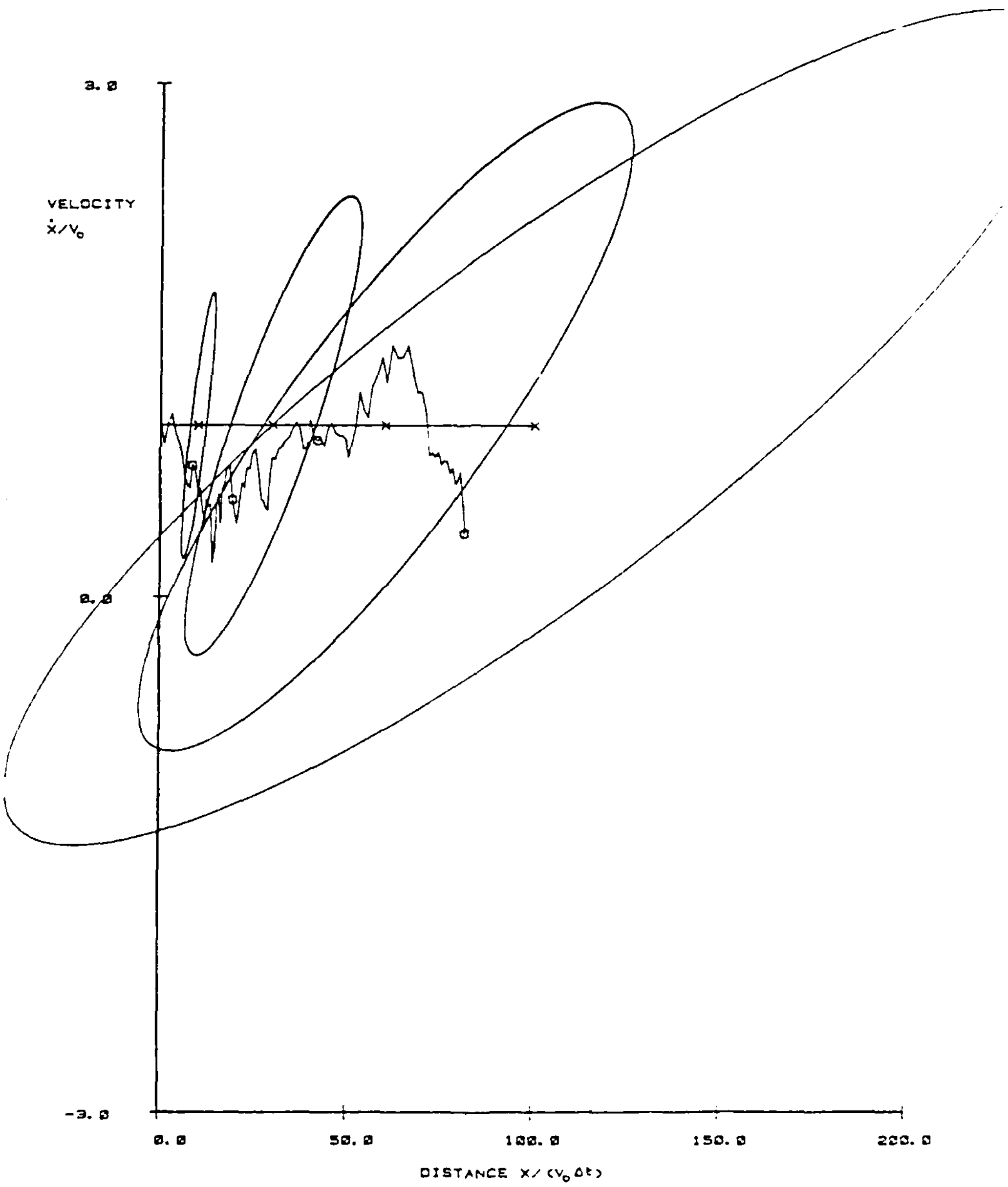


Fig 6 State space trajectory for $q = 10^{-2} \ (V_0/\Delta t)^2$

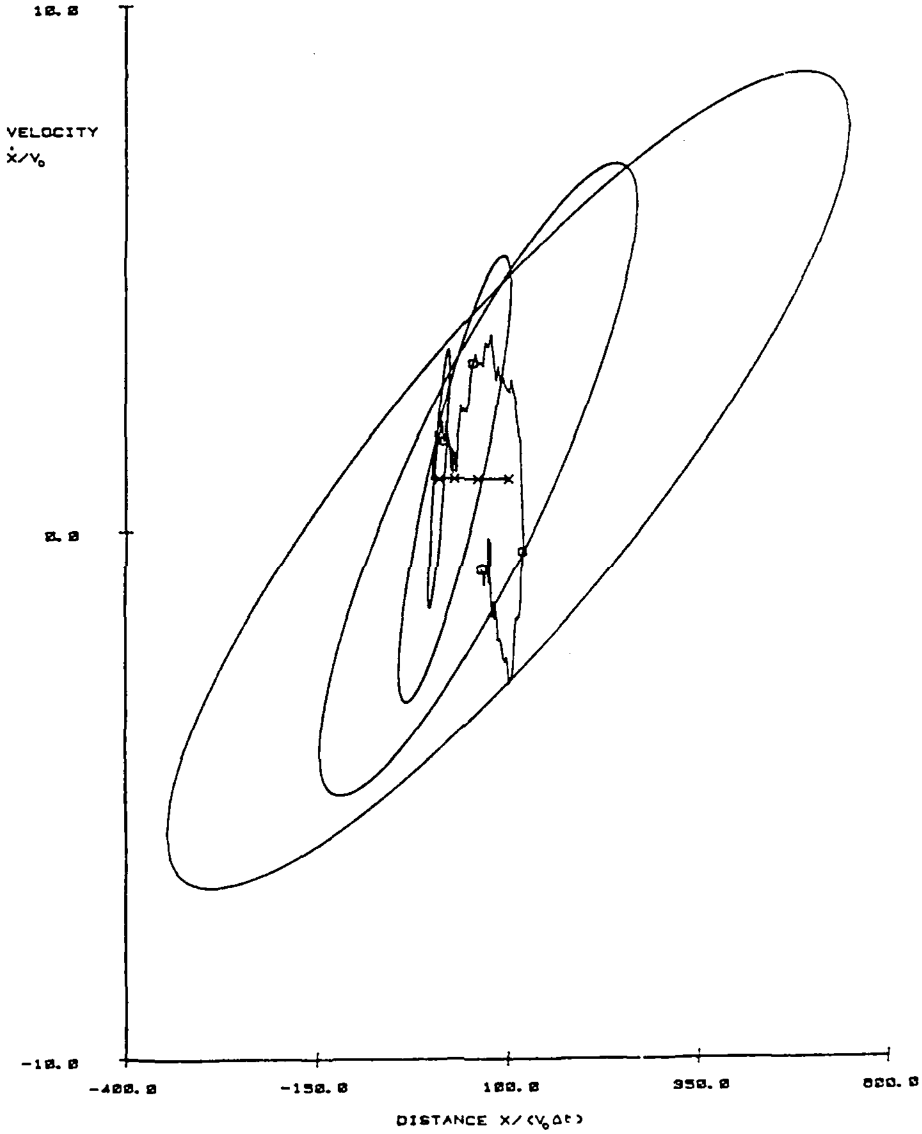
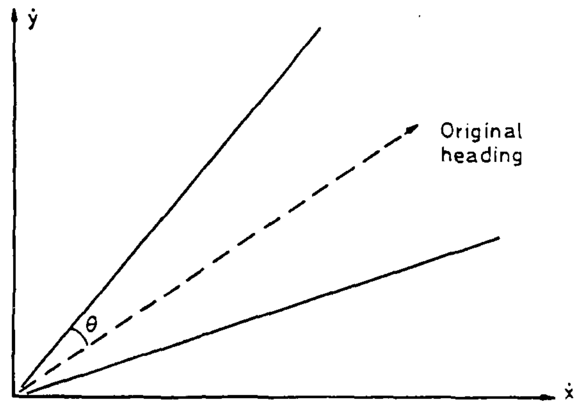


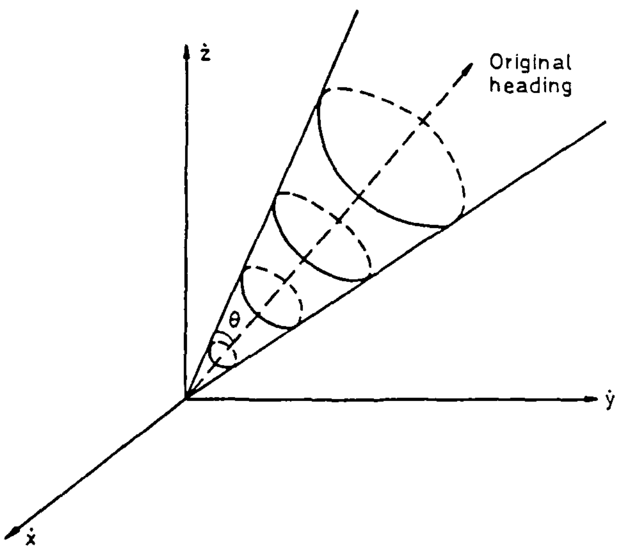
Fig 7 State space trajectory for $q = 10^{-1}\ (V_0/\Delta t)^2$

TR 85071

Fig 8



(a) Two dimensional case



(b) Three dimensional case

Fig 8 Interpretation of target heading change

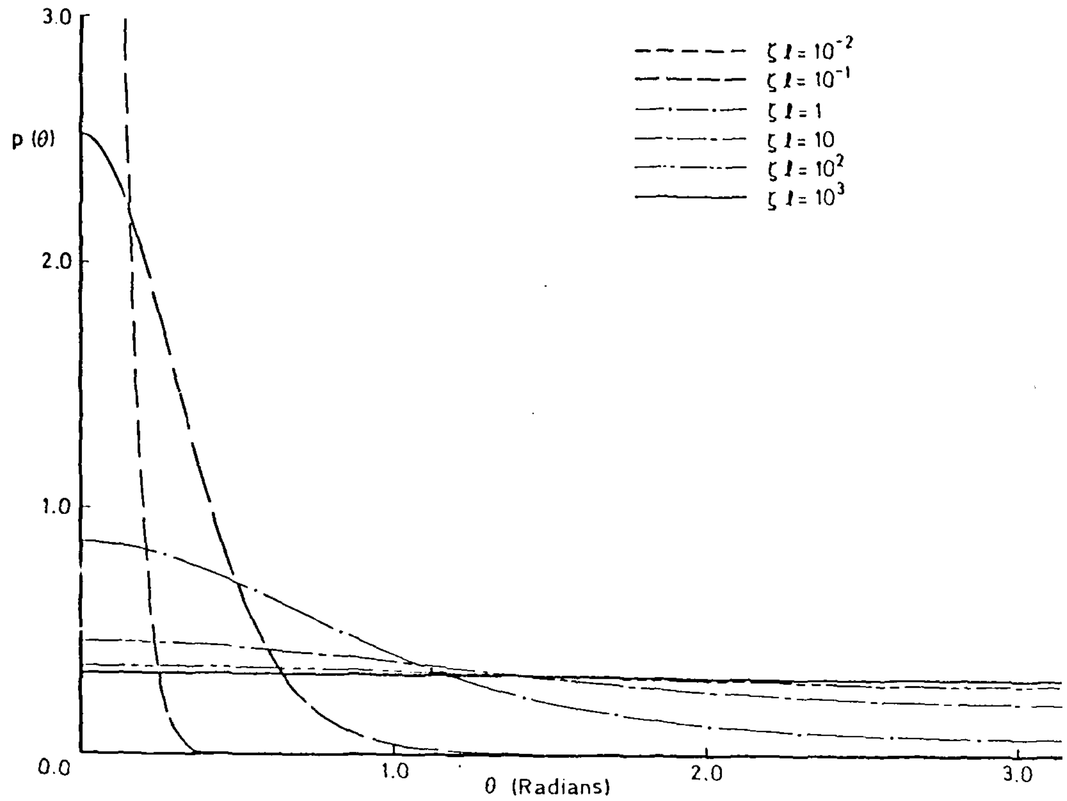


Fig 9 pdf of change in target heading over a period $\ell\Delta t$ for $q = \zeta(V_0/\Delta t)^2$: two-dimensional case

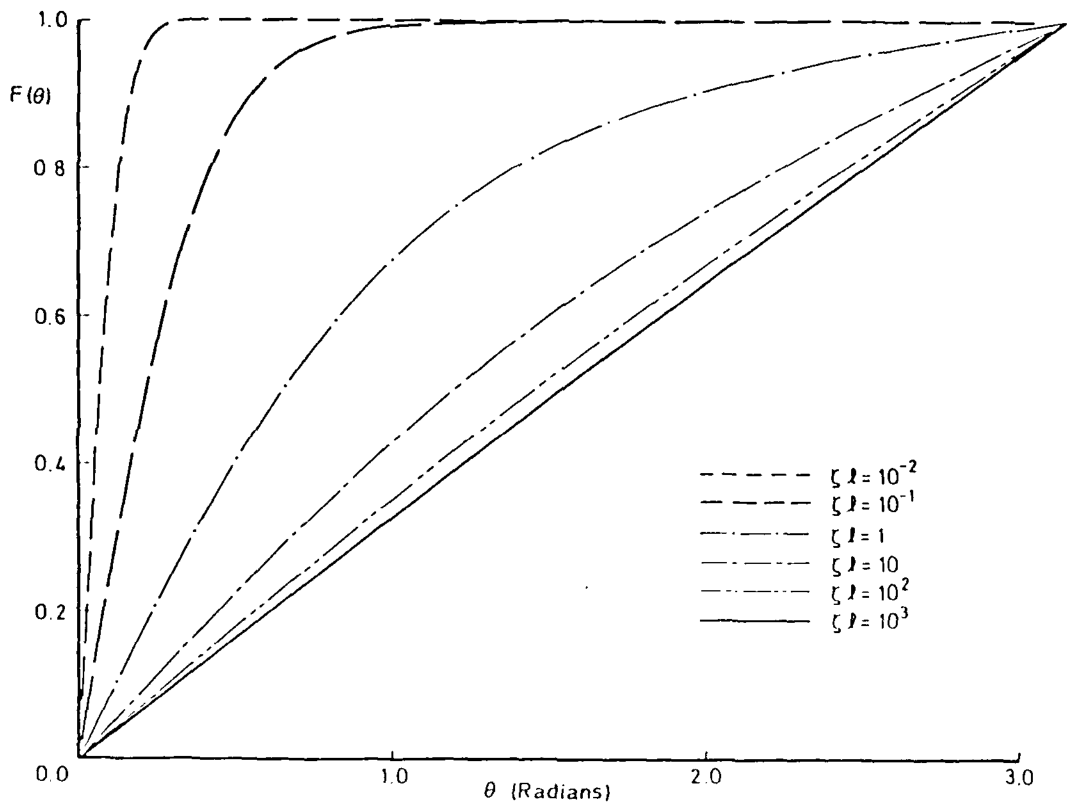
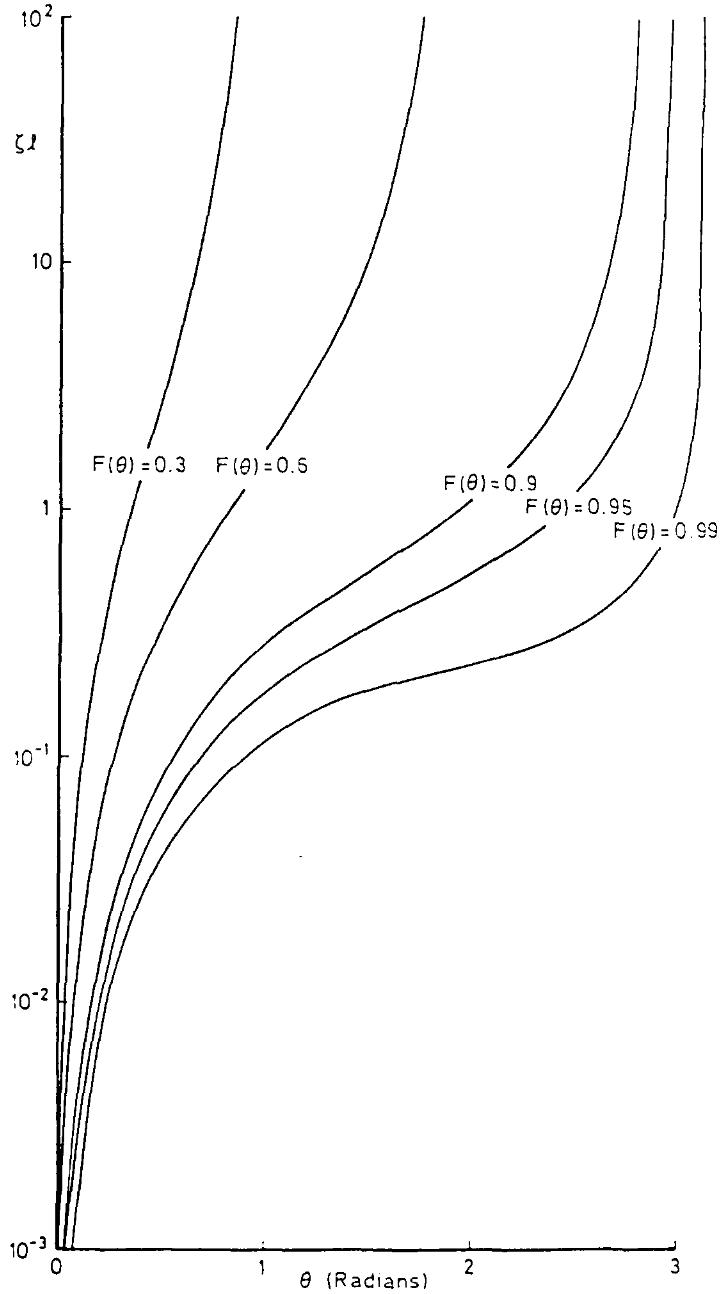


Fig 10 Distribution function of change in target heading over a period $\ell\Delta t$ for $q = \zeta(V_0/\Delta t)^2$: two-dimensional case



TR 85071

Fig 11 Contours of F(θ) as a function of θ and ζℓ: two-dimensional case

Fig 12

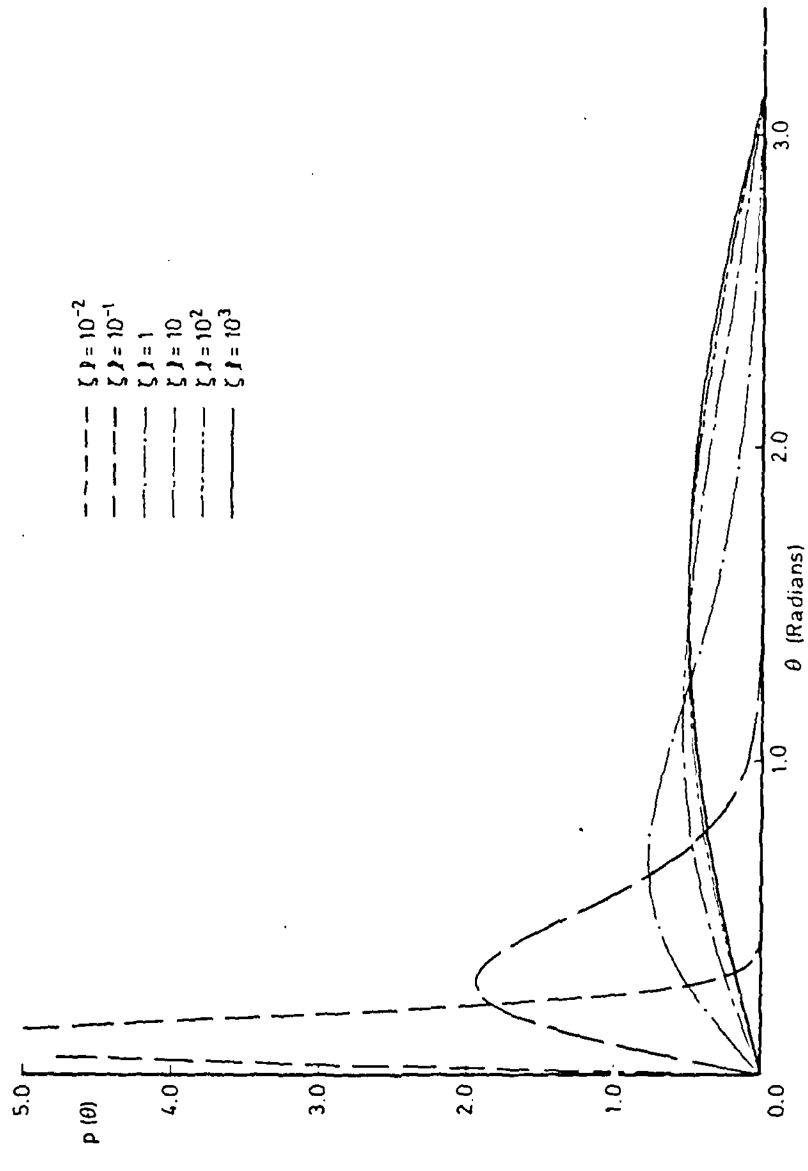


Fig 12 pdf of change in target heading over a period $\ell \Delta t$ for $q = \zeta (V_0/\Delta t)^2$: three-dimensional case

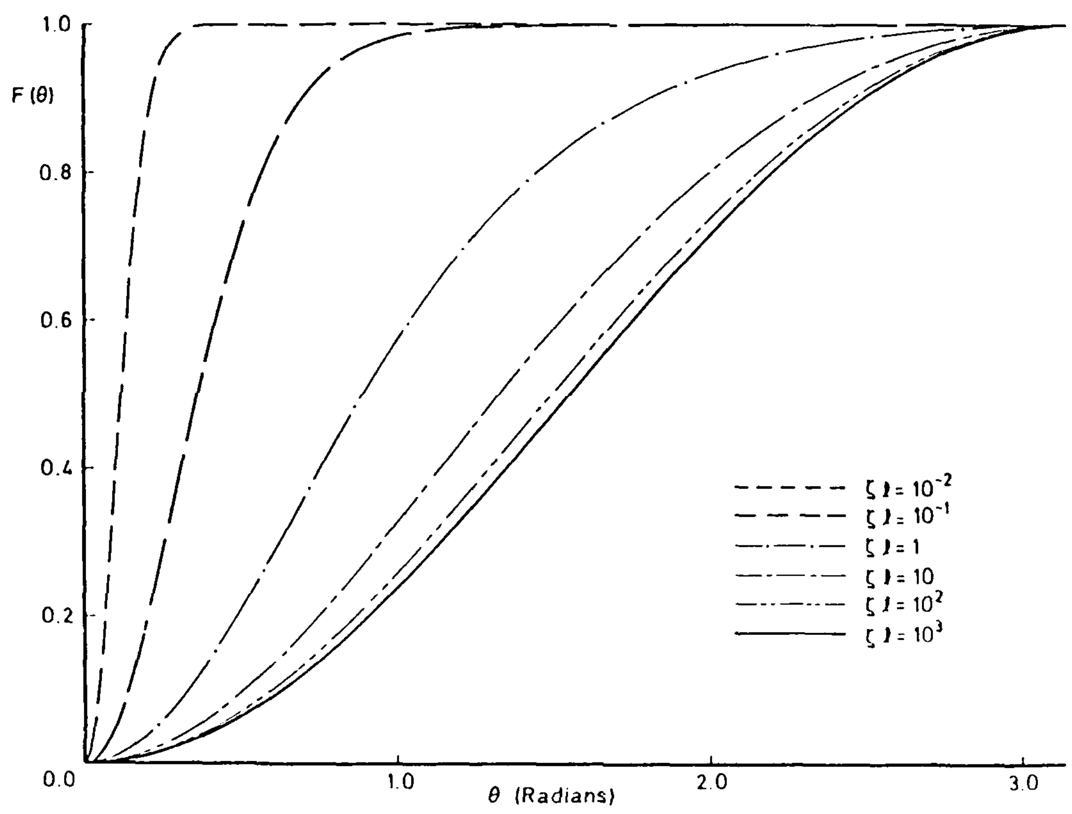


Fig 13 Distribution function of change in target heading over a period $\ell\Delta t$ for $q = \zeta(V_0/\Delta t)^2$: three-dimensional case

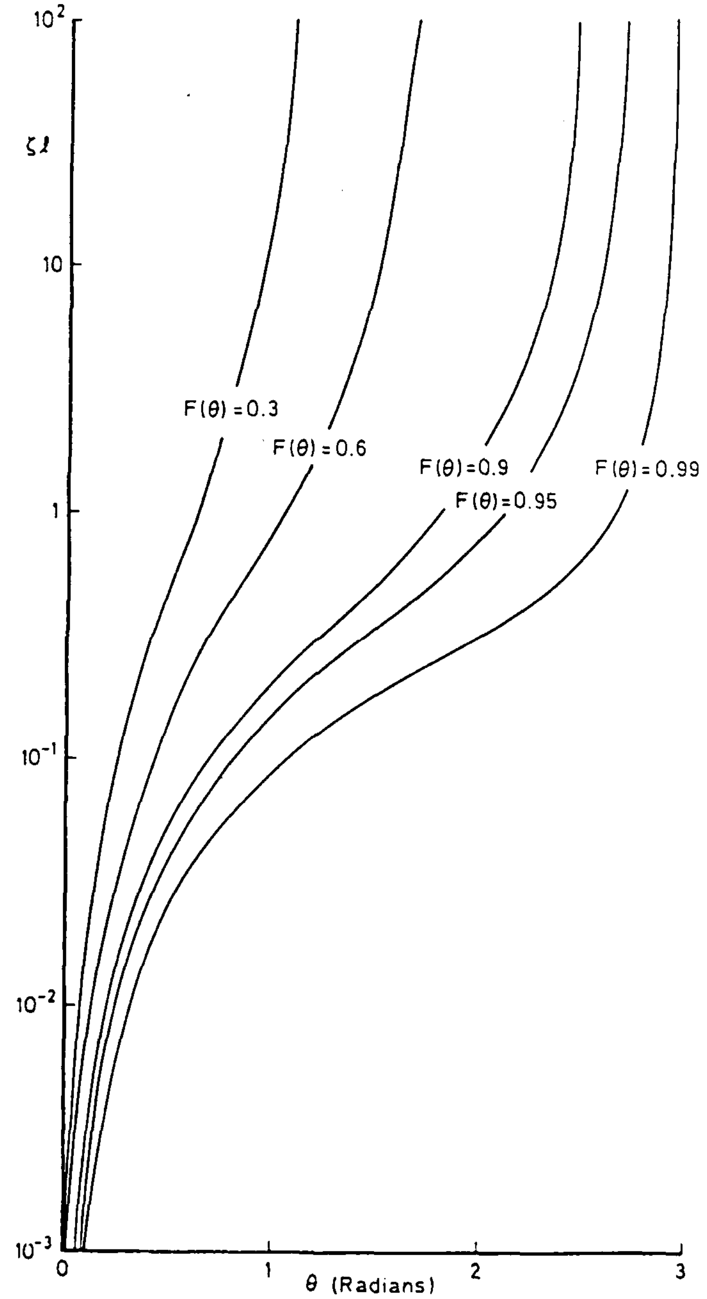


Fig 14 Contours of $F(\theta)$ as a function of θ and $\zeta\ell$: three-dimensional case

TR 85071

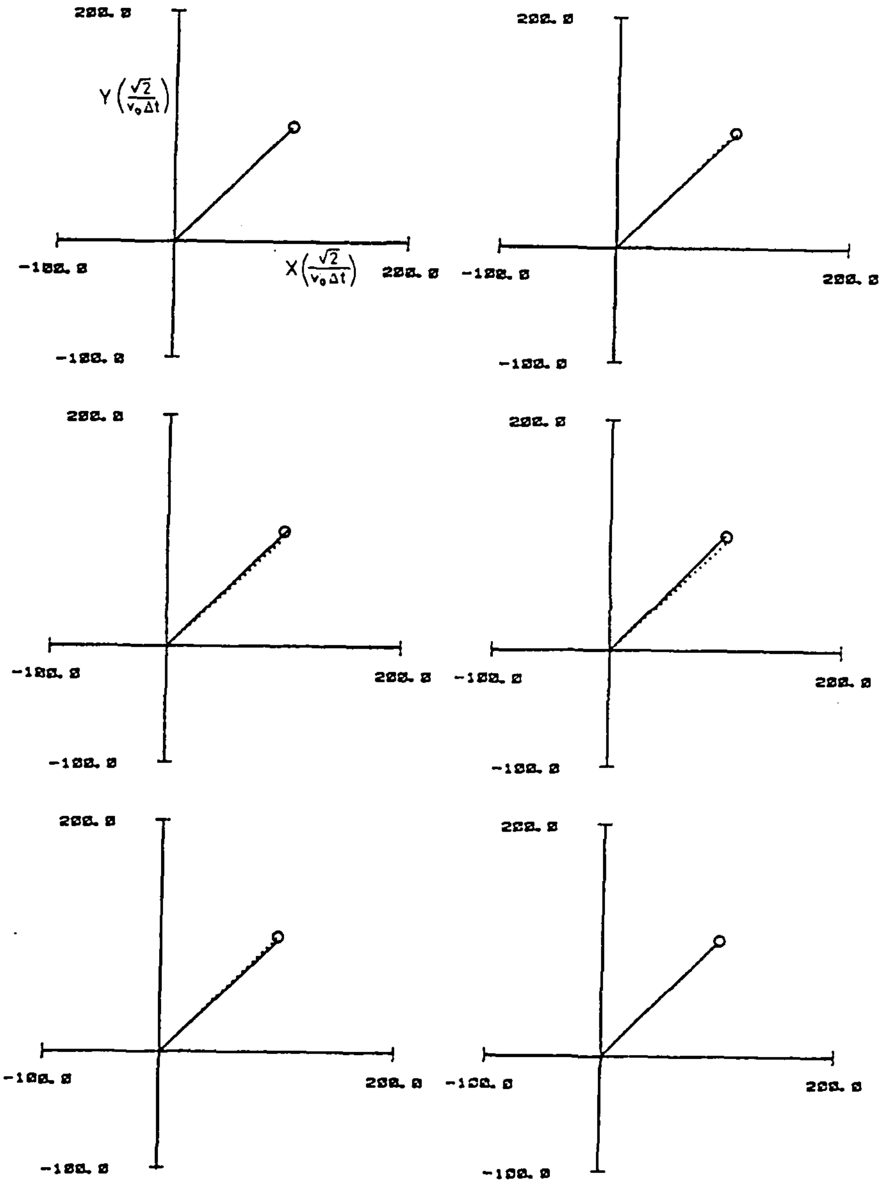


Fig 15 Realizations of two-dimensional trajectories for $q = 10^{-5}\ (V_0/\Delta t)^2$

Fig 16

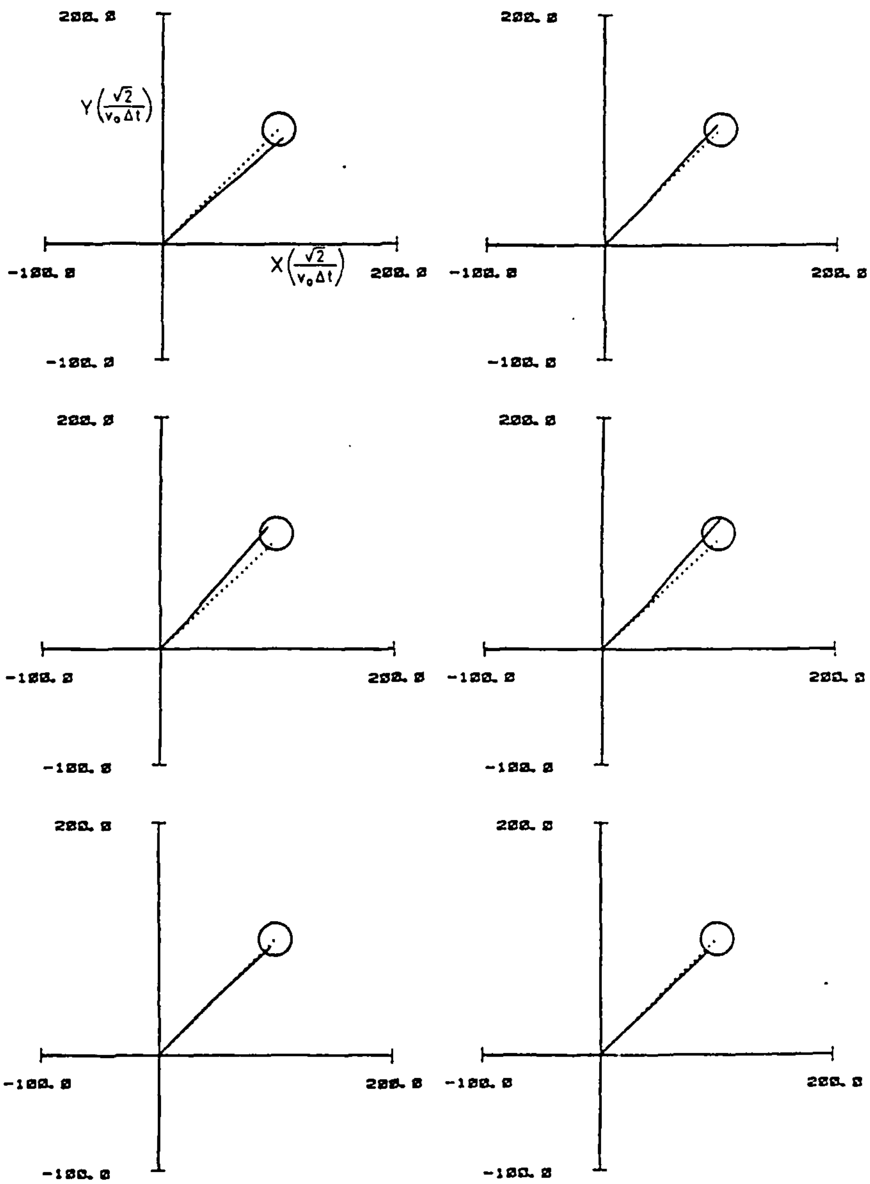


Fig 16 Realizations of two-dimensional trajectories for $q = 10^{-4}\ (V_0/\Delta t)^2$

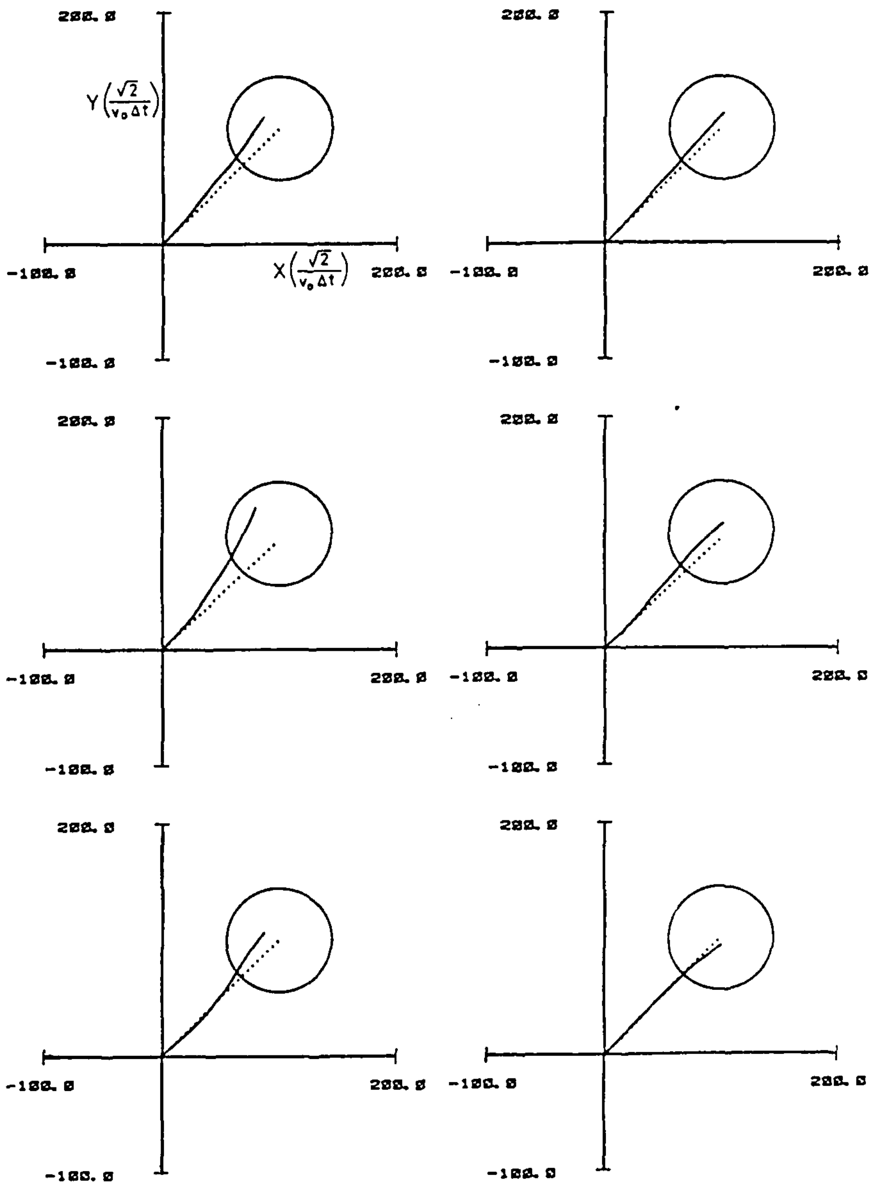


Fig 17 Realizations of two-dimensional trajectories for $q = 10^{-3}\ (V_0/\Delta t)^2$

Fig 18

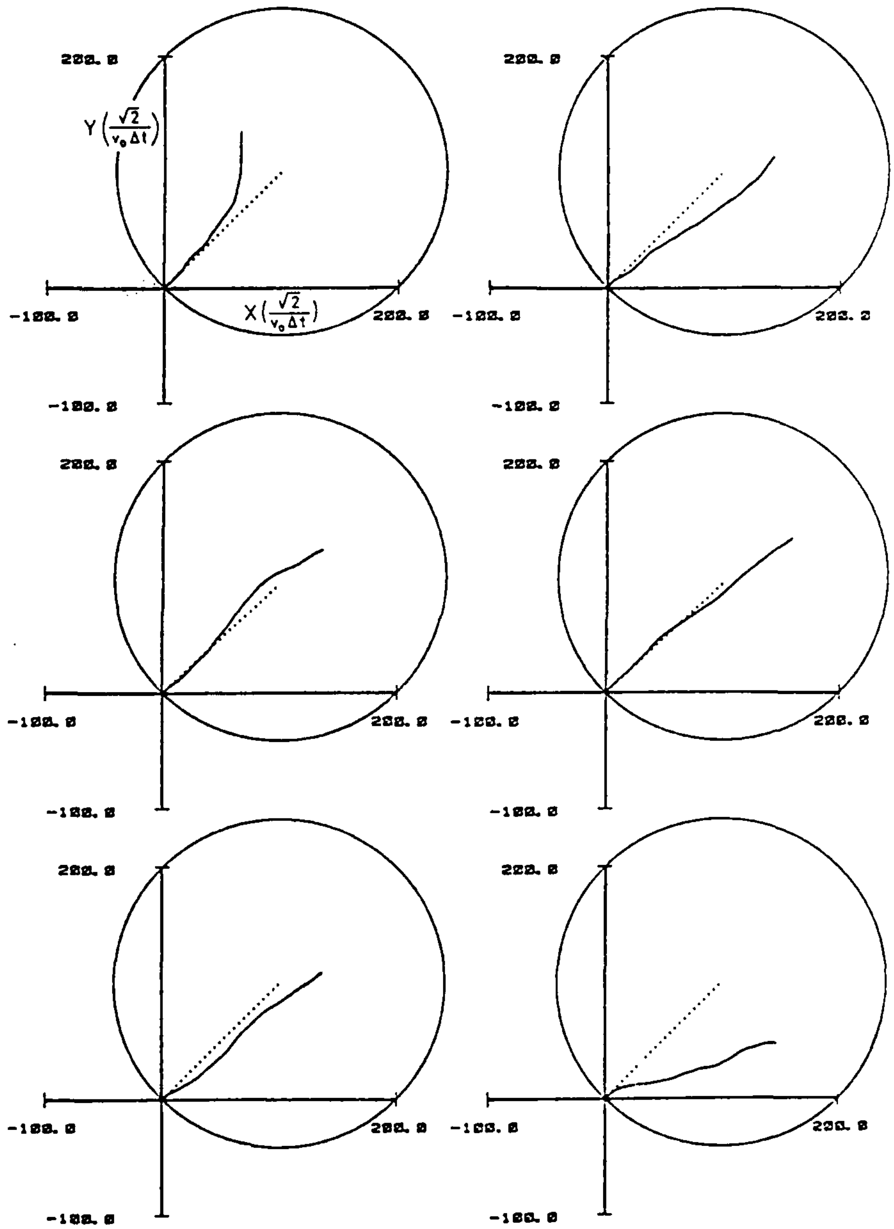


Fig 18 Realizations of two-dimensional trajectories for $q = 10^{-2} \ (V_0/\Delta t)^2$

TR 85071

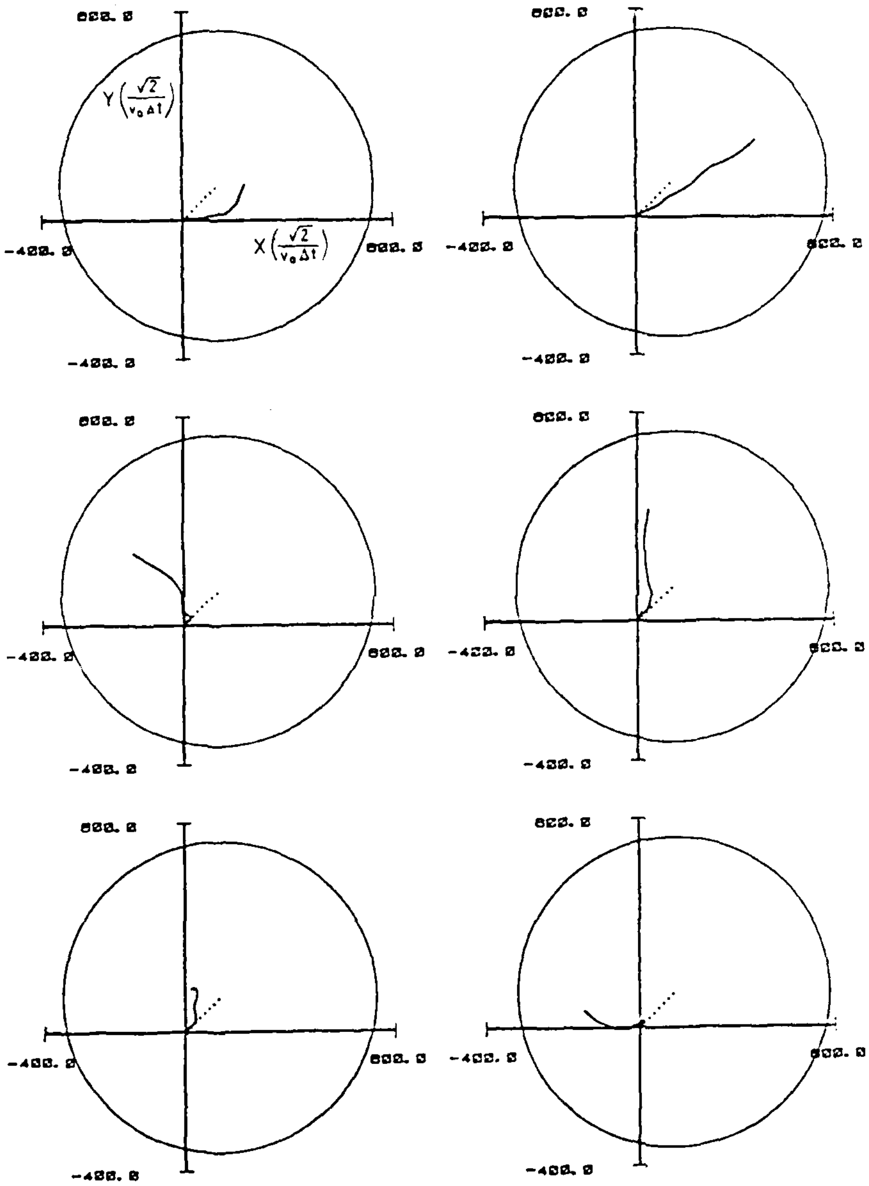
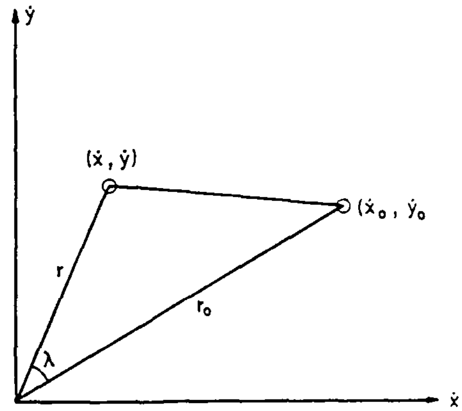
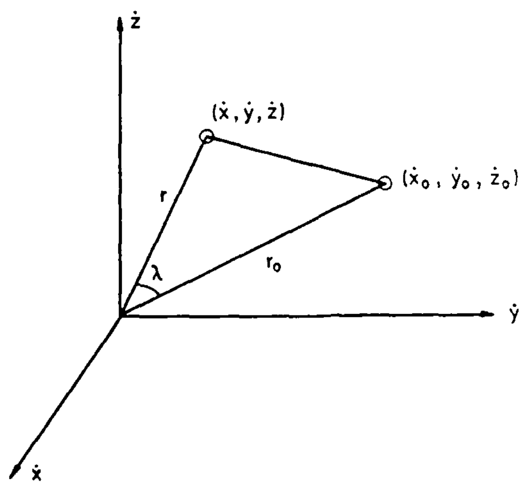


Fig 19 Realizations of two-dimensional trajectories for $q = 10^{-1}\ (V_0/\Delta t)^2$



(a) Two dimensional case



(b) Three dimensional case

Fig B1 Change of target heading

TR 85071

REPORT DOCUMENTATION PAGE

Overall security classification of this page

UNCLASSIFIED

As far as possible this page should contain only unclassified information. If it is necessary to enter classified information, the box above must be marked to indicate the classification, e.g. Restricted, Confidential or Secret.

1. DRIC Reference (to be added by DRIC)	2. Originator's Reference	3. Agency Reference	4. Report Security Classification/Marking
	RAE TR 85071		UNCLASSIFIED

5. DRIC Code for Originator	6. Originator (Corporate Author) Name and Location
7673000W	Royal Aircraft Establishment, Farnborough, Hants, UK
5a. Sponsoring Agency's Code	**6a. Sponsoring Agency (Contract Authority) Name and Location**

7. Title The characteristics of the second-order target model

7a. (For Translations) Title in Foreign Language

7b. (For Conference Papers) Title, Place and Date of Conference

8. Author 1. Surname, Initials	9a. Author 2	9b. Authors 3, 4	10. Date	Pages	Refs.
Salmond, D.J.			August 1985	48	11
11. Contract Number	**12. Period**	**13. Project**	**14. Other Reference Nos.**		
			AW 18		

15. Distribution statement

(a) Controlled by –

(b) Special limitations (if any) –

If it is intended that a copy of this document shall be released overseas refer to RAE Leaflet No.3 to Supplement 6 of MOD Manual 4.

16. Descriptors (Keywords) (Descriptors marked * are selected from TEST)

Tracking filters. Target models.

17. Abstract

Statistics of the trajectories generated by a second-order target model are presented, and using these statistics the distribution of the change in target heading over any time period is derived. This distribution is the basis of a new technique for choosing the model manoeuvre parameter which is suitable for a given class of targets. Design curves for this technique are supplied

Keywords: Great Britain;

RAE Form A143 (revised October 1980)

END

DATE
FILMED
3-87

DTIC